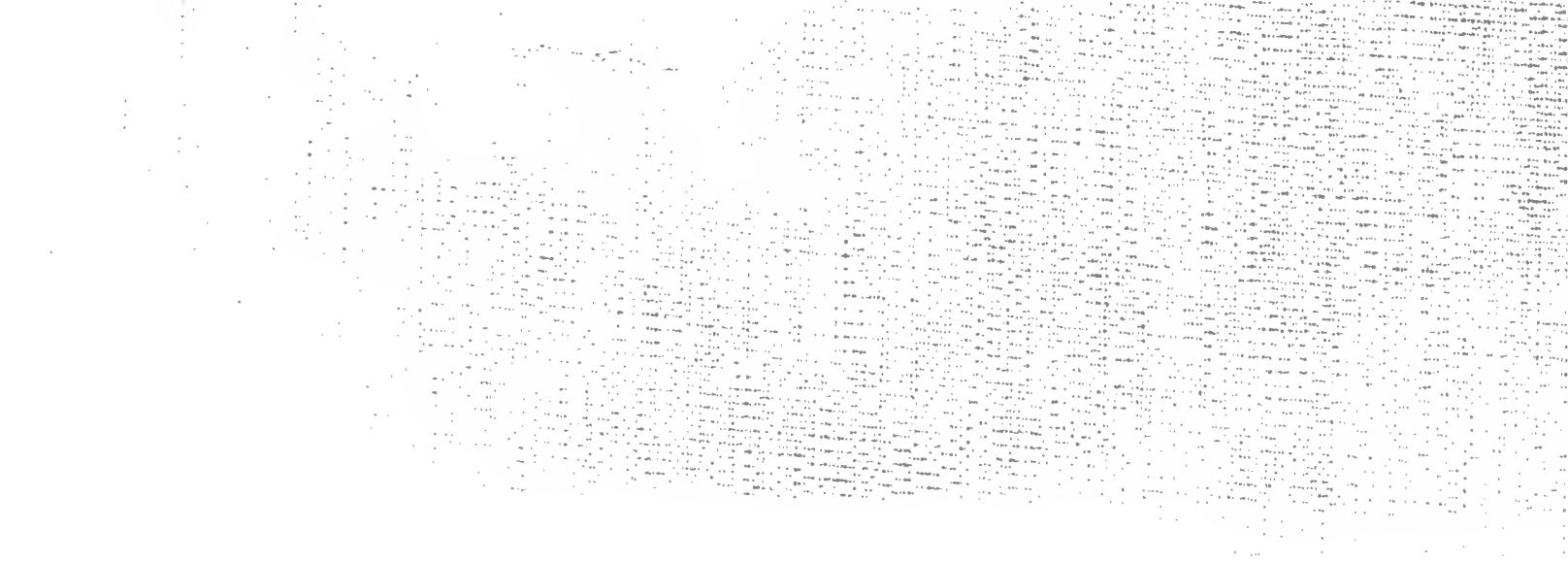

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THEORETICAL AND PRACTICAL GRAPHICS

AN EDUCATIONAL COURSE

ON THE

THEORY AND PRACTICAL APPLICATIONS OF MECHANICAL DRAWING

BY

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CITY

ADVANCE SHEETS ISSUED TO THE CLASS OF '96.

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THE PRINCETON PRESS.

C. S. ROBINSON & CO., UNIVERSITY PRINTERS.

PRINCETON, N. J.

THEORETICAL AND PRACTICAL GRAPHICS.

CHAPTER I.

FIRST PRINCIPLES, WITH GENERAL SURVEY OF THE FIELD OF GRAPHIC SCIENCE.

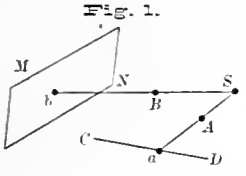
1. Geometrically considered, any combination of points, lines and surfaces is called a *figure*.

A figure lying wholly in one plane is called a *plane figure*; otherwise a *space figure*.

2. Among the methods of investigating and demonstrating the mathematical properties of figures and of solving problems relating to them, that called *projection* is at once one of the most valuable and interesting, constituting, as it does, the common basis of nearly all graphic representations, whether of artist, architect or engineer.

When using this method figures are always considered in connection with a certain point called a *centre of projection*.

Fig. 1.



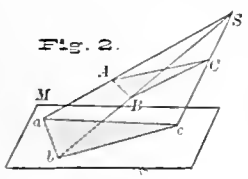
In Fig. 1 let S be an assumed centre of projection and A any point in space. The straight line SA, joining S with A, is called a *projecting line* or *ray*, or simply a *projector*, and its intersection, a, with any line CD, its *projection* upon that line. It is otherwise expressed by saying that A is projected upon CD at a.

In the same way the point B is projected[1] from S upon the plane MN at b; or, in other words, b is—for the assumed position of S—the projection of B upon the plane.

It is with projection upon a plane that we are principally concerned.

The word "projection" is used not only to indicate the method of representation but also the representation itself. In certain other branches of mathematics it has a yet more extended significance, being employed to denote the representation of any curve or surface upon any other.

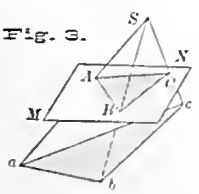
Fig. 2.



3. A figure, as ABC (Fig. 2), is projected upon a plane, MN, by drawing projectors, SA, SB, SC, through its vertices and prolonging them, if necessary[2], to meet the plane. The figure abc, formed by joining the points in which the projectors intersect the plane, is then the projection of the first, or *original* figure.

The plane upon which the projection is made is called the *plane of projection*.

Fig. 3.



4. Were abc (Fig. 3) the original figure and MN the plane of projection, then would ABC be the projection desired. Each figure may thus be considered a projection of the other for a given position of the centre of projection, and when so related figures are said to *correspond* to each other. Points on the same projector, as a and A, are called *corresponding points*.

[1] Were S the muzzle of a gun, and B a bullet speeding from it toward the plane, it would be projected against or through the plane at b. The appropriateness of the term "projection" is obvious.

[2] In Fig. 3 the projectors meet the plane between the centre S and the given figure.

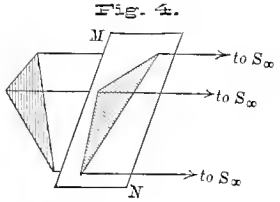
5. Having indicated what projections are and how obtained, it will be well, before giving their grand divisions and sub-divisions, to state the nature and extent of the field in which they may be employed.

The mathematical properties of geometrical figures, as also the propositions and problems involving them, are divided into two classes, *metrical* and *descriptive*. In the first class the idea of *quantity* necessarily enters, either directly—as in measurement, or indirectly—as in ratio[1]. In the second or descriptive class, however, we find involved only those properties dependent upon *relative position*.[2] Descriptive properties are unaltered by projection while but few metrical properties are projective. The province of projection is obvious.

6. Descriptive Geometry is that branch of mathematics in which figures are represented and their descriptive properties investigated and demonstrated by means of projection.

DIVISIONS OF PROJECTIONS.

Fig. 4.

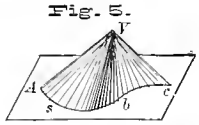


7. All projections may be divided into two general classes, *Central* and *Parallel*.

If the centre of projection be at a finite distance, as in Figs. 2 and 3, the projection obtained is called a *central projection*; but if we suppose it to be at infinity, as in Fig. 4, projectors from it will then evidently be parallel, and the resulting figure is called a *parallel projection* of the original figure. Parallel projection is thus seen to be merely a special case of central projection, yet each has been independently developed to a high degree and has an extensive literature.

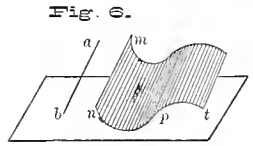
8. The terms *Conical* and *Cylindrical* are employed by many writers synonymously with central and parallel respectively. Central projections are also occasionally called *Radial* or *Polar*.

Fig. 5.



REMARK.—A straight line is said to generate a conical surface (see Fig. 5) when it constantly passes through a fixed point (the *vertex*), and is guided in its motion by a given fixed curve (the *directrix*). The moving straight line is a *generatrix* of the surface, and its various positions are called *elements* of the surface.

Fig. 6.



If the vertex of a conical surface be removed to infinity the elements will become parallel and we shall have a *cylindrical* surface, which may be also defined as the surface (see Fig. 6) generated by a straight line that is guided in its motion by a given fixed curve, and is, in any position, parallel to a given fixed straight line.

The origin of the terms *conical* and *cylindrical* as applied to projection is obvious.

We have now to mention the more important sub-divisions of projections, with the sciences based upon them. The names depend in certain cases upon the nature of the centre of projection while in others they are due to some particular application.

Under *Central* (or *Conical*) *projection* we have:—

9. *Projective Geometry* (*Geometry of Position*). While in its most general sense this science includes all central projections (and therefore those of Articles. 10–13) yet in its ordinary acceptation it may be defined as that branch of mathematics in which—*with the centre of projection considered as a mathematical point at a finite distance from the line or plane of projection*—the descriptive properties of geometrical figures are investigated and established.

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[1] The following will illustrate under *metrical*:
(a). The lateral area of a cylinder is equal to the product of the perimeter of its right section by an element of the surface.
(b). Two tetrahedrons which have a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the three edges of the equal trihedral angles.

[2] Examples under the *descriptive* class:
(a). If a line is perpendicular to a plane, any plane containing the line will also be perpendicular to the plane.
(b). To locate the centre of a sphere whose surface shall contain four definitely located points in space.
(c). To determine the form of the curve of intersection of two given arches.

Its chief practical application is in *Graphical Statics* in which the stresses in bridge and roof trusses or other engineering constructions are determined graphically by means of diagrams.

10. *Perspective.—If the centre of projection is the eye* of the observer, the projection is called a *perspective* or *scenographic projection*, or—more commonly—simply a *perspective*. The plane of projection is then called the *perspective plane* or *picture plane* and is always vertical. The position of the eye is called the *point of sight* or *station point*, and the projectors are called *visual rays*.

Applied in the graphical construction usually preliminary to art work in water colors or oil; also in architectural perspectives and in scientific illustrations of machinery, etc.

It may be remarked that *any projection, central or parallel, presents to the eye the same appearance as the figure projected would if viewed from the centre of projection.*

11. *Relief-perspective.*—This differs from the perspective just defined in requiring, in addition to the usual perspective plane, a second plane parallel to it called a *vanishing plane*, the required representation appearing *in relief* between the two planes—a solid perspective, so to speak.

Employed chiefly in the construction of bas-reliefs and theatre decorations.

12. *Shadows (artificial light).*—If the centre of projection is an artificial light, as the electric or that of a candle—either of which may, without appreciable error, be treated in graphical constructions as a mere point—the projectors will be *rays of light* and the projection will be the *shadow* of the figure projected.

Employed in obtaining shadow effects in interiors.

13. *Photogrammetry* or *Photometrography*, the application of photography to surveying, the centre of projection becoming the optical centre of the lens.

14. Under *Parallel (or Cylindrical) projection* we have:—

(a). *Oblique* or *Clinographic*, and

(b). *Perpendicular* or *Orthographic*, also called *Orthogonal* or *Rectangular*.

These divisions are based upon the direction of the projectors with respect to the plane of projection, they being—as the names imply—inclined to it in oblique projection and perpendicular to it in orthographic.

OBLIQUE PROJECTION.

15. The shadow of an object in the sunlight would be its oblique projection, the sun's rays being practically parallel.

16. Oblique projection is usually called *Clinographic* when employed in Crystallography.

17. In its other applications, when not simply called oblique, this projection is variously termed *Cavalier Perspective*, *Cabinet Projection* and *Military Perspective*—the plane of projection being vertical in the first and second while in the last it is horizontal.

Oblique projection gives a pictorial effect closely analogous to a true perspective, yet is far more simple in its construction, and is much used for showing the form or method of assemblage of parts, or *details*, of machinery and architectural work.

ORTHOGRAPHIC PROJECTION, UPON A SINGLE PLANE.

18. When but one plane of projection is employed the only important applications of orthographic projection having special names are—

(a). *One-Plane Descriptive*, otherwise called *Horizontal Projection*.

Employed chiefly in fortification and general topographical work, in which the lines and surfaces represented are mainly horizontal.

(b). *Axonometric* (including *Isometric*) *Projection*.

Has the same range of application as oblique projection, viz., to objects whose lines lie mainly in directions mutually perpendicular to each other.

ORTHOGRAPHIC PROJECTION UPON MUTUALLY PERPENDICULAR PLANES.

19. When upon two (or more) mutually perpendicular planes orthographic projection becomes the *Géometrie Descriptive* of Gaspard Monge, who reduced its principles to scientific form in the latter part of the eighteenth century.

The tendency—a logical one—toward the general adoption of the title "Descriptive Geometry" in the broad sense of Art. 6 would make it seem advisable to appropriate the name *Monge's Descriptive* to this—the most important division of graphic science, that we may find in it a hint as to its source and at the same time pay to its inventor the honor of perpetual association of his name with his creation. As originally defined by Monge it is the application of orthographic projection,

(a) to the exact representation upon a plane surface, as that of a drawing-board, of all objects capable of rigorous definition, and

(b) to the solution of problems relating to these objects in space and involving only their properties of form and position.

It might with propriety be divided into *pure* and *applied*, the former being the abstract science, in which the mathematical relations existing between figures and their projections are examined and applied in the solution of certain fundamental problems of the point, line and plane; while the latter division would naturally include the application of these principles and methods to the solution of problems relating to the various elementary and higher mathematical surfaces and to machine drawing and design, shades, shadows, perspective, stone-cutting, spherical projections, crystallography, pattern-making, carpentry, etc.

ADDITIONAL REMARKS ON NOMENCLATURE.

The student may find the following serviceable by way of enabling him to get clear ideas of the distinctions between various systems.

The term *Geometry*, unqualified, is usually understood to refer to the synthetic method of investigation of the form, position, ratio and measurement of geometrical figures, the reasoning being from particular to general truths by the aid of diagrams. In contra-distinction to other geometries it is frequently called *Euclidean*, after the celebrated Greek geometer, Euclid, (about 330–275 B. C.) who organized its theorems and problems into a science.

In *Coördinate* (or *Analytical*) Geometry the figure considered is referred to a system of coördinates and the relation existing between the coördinates of every point of the figure is expressed by means of an equation in which the coördinates are represented by algebraic symbols. The operations performed are algebraic and the method of reasoning from general to particular truths.

Although the invention of Analytical Geometry has been attributed to Descartes it is now recognized that he neither originated the use of coördinates nor the representation of curves and surfaces by means of equations. As the first to give complete scientific form to the analytic method his name has, however, justly been given to the most important division of coördinate geometry, *Cartesian*. But the writers of the present day under that head do not by any means confine themselves to the system of coördinates employed by him, which consisted of intersecting straight lines usually perpendicular to each other.

In selecting the title "Projective Geometry" for the science defined in Art. 9, the eminent Cremona says, "I prefer not to adopt that of *Higher Geometry*, (*Géometrie superieure, höhere Geometrie*) because that to which the title 'higher' at one time seemed appropriate, may to-day have become very elementary; nor that of *Modern Geometry* (*neuere Geometrie*) which in like manner expresses a merely relative idea, and is moreover open to the objection that although the methods may be regarded as modern, yet the matter is to a great extent old. Nor does the title *Geometry of Position* (*Geometrie der Lage*) as used by Staudt seem to me a suitable one, since it excludes the consideration of the metrical properties of figures. I have chosen the name of *Projective Geometry* as expressing the true nature of the methods, which are based essentially upon central projection or perspective. And one reason which has determined this choice is that the great Poncelet, the chief creator of the modern methods, gave to his immortal book the title of "*Traité des propriétés projectives des figures. (1822).*"

Cremona further states that "there is one important class of metrical properties (anharmonic properties) which are projective, and the discussion of which therefore finds a place in the Projective Geometry." But the positional definition since given by Staudt for the anharmonic ratio of four points, which removes these properties from the class metrical to the class descriptive (which last are always projective), to that extent justifies the title employed by him, while making Cremona's choice none the less a fortunate one.

There are other geometries, belonging to what may be called speculative mathematics, based upon "quasi-geometrical notions, those of more-than-three-dimensional space, and of non-Euclidean two-and-three-dimensional space, and also of the generalized notion of distance."*

The following will illustrate a method of arriving at a conception of non-Euclidean two-dimensional geometry. "Imagine the earth a perfectly smooth sphere. Understand by a plane the surface of the earth, and by a line the apparently straight line (in fact an arc of a great circle) drawn on the surface. What experience would in the first instance teach would be Euclidean two-dimensional geometry; there would be intersecting lines, which, produced a few miles or so, would seem to go on diverging, and apparently parallel lines which would exhibit no tendency to approach each other; and the inhabitants might very well conceive that they had by experience established the axiom that two straight lines cannot enclose a space, and also the axiom as to parallel lines. A more extended experience and more accurate measurements would teach them that the axioms were each of them false; and that any two lines, if produced far enough each way, would meet in two points: they would, in fact, arrive at a spherical geometry accurately representing the properties of the two-dimensional space of their experience. But their original Euclidean geometry would not the less be a true system; only it would apply to an ideal space, not the space of their experience."*

* Cayley.

CHAPTER II.

ARTISTIC AND TECHNICAL FREE-HAND DRAWING.—SKETCHING FROM MEASUREMENT.—FREE-HAND LETTERING.—CONVENTIONAL REPRESENTATIONS.

20. Drawings, if classified as to the *method of their production*, are either *free-hand* or *mechanical;* while as to *purpose* they may be *working drawings*, so fully dimensioned that they can be worked from and what they represent may be manufactured; or *finished* drawings, illustrative or artistic in character and therefore shaded either with pen or brush and having no hidden parts indicated by dotted lines as in the preceding division. Finished drawings also lack figured dimensions.

Working drawings of parts or "details" only of a structure are called *detail drawings*; and the representation of a structure as a whole with all its details in their proper relative position, hidden parts indicated by dotted lines, etc., is termed a *general* or *assembly* drawing.

21. While mechanical drawing is involved in making the various essential views—plans, elevations and sections—of all engineering and architectural constructions and in solving the problems of *form* and *relative position* arising in their design, yet to the engineer the ability to sketch effectively and rapidly, *free-hand*, is of scarcely less importance than to handle the drawing instruments skilfully; while the success of an architect depends in still greater measure upon it.

We must distinguish, however, between *artistic* and *technical* free-hand work. The architect must be master of both; the engineer necessarily only of the latter.

To secure the adoption of his designs the architect relies largely upon the effective way in which he can finish, either with pen and ink or in water-colors, the perspectives of exterior and interior views; and such drawings are judged mainly from the artistic standpoint. While it is not the province of this treatise to instruct in such work a word of suggestion may properly be introduced for the student looking forward to architecture as a profession. He should procure Linfoot's *Picture Making in Pen and Ink*, Miller's *Essentials of Perspective* and Delamotte's *Art of Sketching from Nature;* and with an experienced architect or artist, if possible, but otherwise by himself, master the principles and act on the instructions of these writers.

22. Since the camera makes it, fortunately, no longer essential that a civil engineer should be a landscape artist as well, his free-hand work has become more restricted in its scope and more rigid in its character, and like that of the machine designer it may properly be called *technical*, from its object. Yet to attain a sufficient degree of skill in it for all practical and commercial purposes is possible to all and among them many who could never hope to produce artistic results. It is confined mainly to the making of *working sketches*, *conventional representations* and *free-hand lettering*, and the equipment therefor consists of a pencil of medium grade as to hardness; lettering pens—Falcon or Gillott's 303, with Miller Bros. "Carbon" pen, No. 4; either a note-book or a sketch-block or pad, and—for sketching from measurement—a two-foot pocket-rule; calipers, both external and internal, for taking outside and inside diameters; a pair of pencil compasses for making an occasional circle too large to be drawn absolutely free-hand; and a steel tape-measure for large work if one can have assistance in taking notes, but otherwise a long rod graduated to eighths.

23. In the evolution of a machine or other engineering project the designer places his ideas on paper in the form of rough and mainly free-hand sketches, beginning with a general outline, or "skeleton" drawing of the whole, on as large a scale as possible, then filling in the details, separate —and larger—drawings of which are later made to exact scale. While such preliminary sketches are not drawn literally "to scale" it is obviously desirable that something like the relative proportions should be preserved and that the closer the approximation thereto the clearer the idea that they will give to the draughtsman or workman who has to work from them. A habit of close observation must therefore be cultivated, of analysis of form and of relative direction and proportion, by all who would succeed in draughting, whether as designers or merely as copyists of existing constructions. While the beginner belongs necessarily in the latter category he must not forget that his aim should be to place himself in the ranks of the former both by a thorough mastery of the fundamental theory that lies back of all correct design and by such training of the hand as shall facilitate the graphic expression of his ideas. To that end he should improve every opportunity to put in practice the following instructions as to

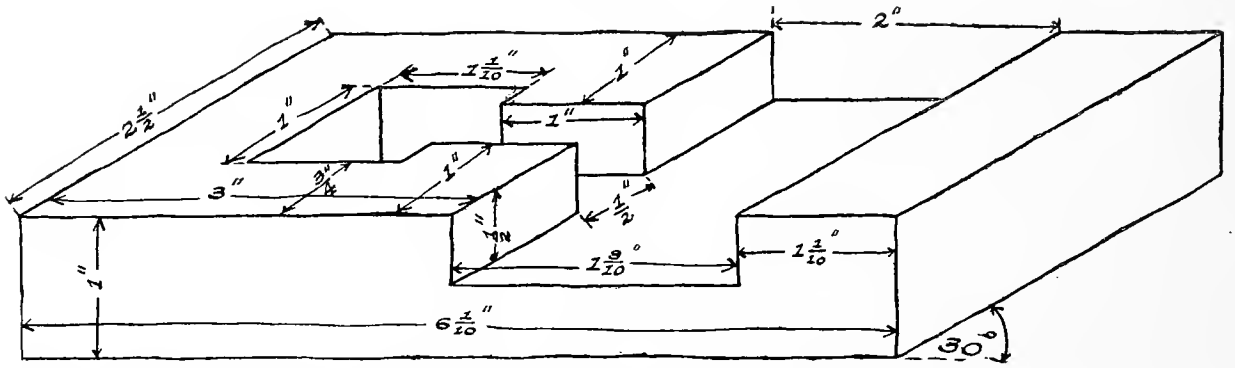
SKETCHING FROM MEASUREMENT,

as each structure sketched and measured will not only give exercise to the hand but also prove a valuable object lesson in the proportioning of parts and the modes of their assemblage.

A free-hand sketch may be as good a working drawing as the exactly scaled—and usually *inked*—drawing that is generally made from it to be sent to the shop.

While several views are usually required yet for objects of not too complicated form, and whose lines lie mainly in mutually perpendicular directions, the method of representation illustrated by Fig. 7, is admirably adapted and obviates all necessity for additional sketches.

Fig. 7



It is an oblique projection (Art. 17) the theory of whose construction will be found in a subsequent chapter, but with regard to which it is sufficient at this point to say that the right angles of the front face are seen in their true form while the other right angles are shown either of 30°, 60°, or 120°; although almost any oblique angle will give the same general effect and may be adopted. Lines parallel to each other on the object are also parallel in the drawing.

Draw first the front face, whose angles are seen in their true form; then run the oblique lines off in the direction which will give the best view.

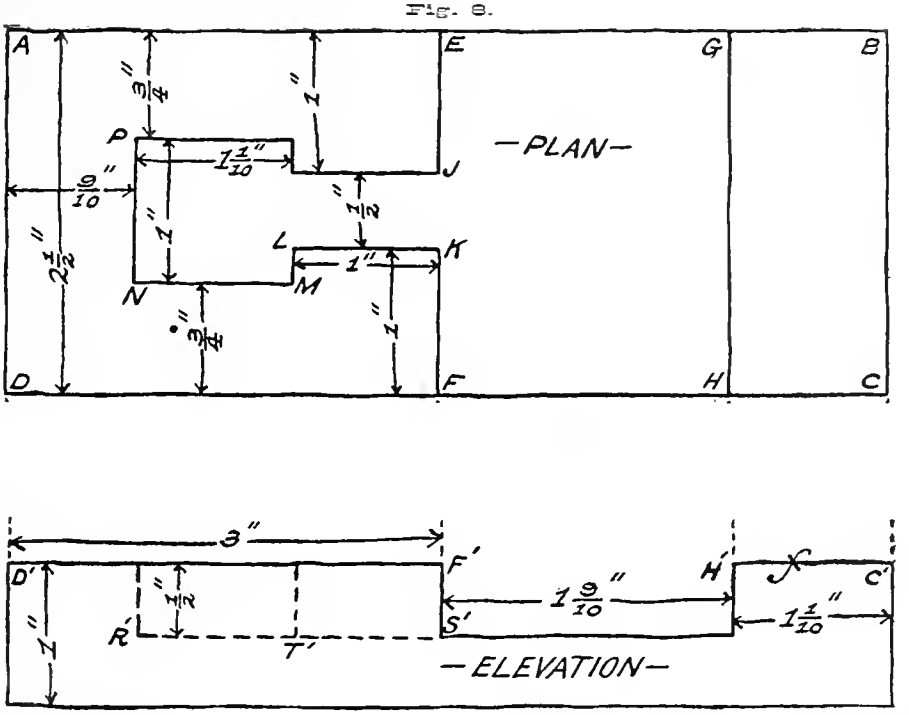
24. While Fig. 7 gives almost the pictorial effect of a true perspective and the object requires no other description, yet for complicated and irregular forms it gives place to the plan-and-elevation mode of representation, the plan being a *top* and the elevation a *front* view of the object. And if two views are not enough for clearness as many more should be added as seem necessary, including what are called *sections*, which represent the object as if cut apart by a plane, separated and a

view obtained perpendicular to the cutting plane, showing the internal arrangement and shape of parts.

In Fig. 8 we have the same object as in Fig. 7, but represented by the method just mentioned. The front view (elevation) is evidently the same in both figures, except that Fig. 8 has dotted lines to indicate the recess which is in full sight in Fig. 7.

The view of the top is placed *at the top* in conformity to the now quite general practice as to location, viz., grouping the various sketches about the elevation, so that the view of the left end is *at the left*, of the right *at the right*, etc.

Fig. 8.



In these views, which fall under Art. 19 as to theoretical construction, entire surfaces are *projected* as straight lines, as *G B C H* in the straight line *H' C'*. Were this a metallic surface and "*finished*" or "*machined*" to smoothness, as distinguished from the surface of a rough casting, that fact would be denoted by an "*f*" *on* the line *H' C'* which represents the entire surface, the cross-line of the "*f*" cutting the line of the surface obliquely, as shown.

25. In sketching, centre-lines and all important centres should be located first and measurements taken from them or from finished surfaces.

Feet and inches are abbreviated to "Ft.," and "In.," as 4 Ft. 6¾ In; also written 4′ 6¾″ and occasionally 4 Ft. 6¾." A dimension should not be written as an improper fraction, $\frac{13}{2}$″ for example, but as a mixed number, $1\frac{5}{2}$″. Fractions should have *horizontal* dividing lines.

Not only should dimensions of successive parts be given but an "over-all" dimension which, it need hardly be said, should sustain the axiom regarding the whole and the sum of its parts.

Dimensions should read in line with the line they are on and either from the bottom or the right hand.

The arrow tips should touch the lines between which a distance is given.

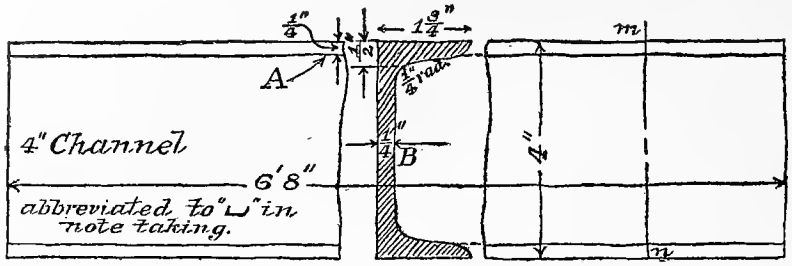
An opening should be left in the dimension line for the figures.

Extension lines should be drawn and the dimension given outside the drawing whenever such course will add to the clearness.

In case of very small dimensions the arrow tips may be located outside the lines, as in Fig. 9, and the dimension indicated by an arrow as at *A* or inserted as at *B* if there is room.

Should a piece of *uniform cross-section* be too long to be represented in its proper relative length on the sketch, it may be broken as in Fig. 9 and the *form* of the section (which in the case supposed will be the same as an *end view*) may be inserted with its dimensions, as in the shaded figure.

Fig 9.



The same dimension should not appear on each view, but each dimension must be given at least once on some view.

In riveted work the "pitch" of the rivets, i. e., their distance from centre to centre ("c. to c.") should be noted, as also that between centre lines or rows, and of the latter from main centre lines. Similarly for bolts or holes. If the latter are located in a circle note the diameter of the circle containing their centres. Note that the hole for a rivet is usually about one-half the diameter of the forged head. In measuring nuts take the width between parallel sides ("width across the flats") and abbreviate for the shape, as "sq.," "hex.," "oct."

For a piece of cylindrical shape a frequently used symbol is the circle, as 4″ O for 4″ diameter; but it is even clearer to use the abbreviation of the latter word, viz., "diam."

In taking notes on bolts and screws the outside diameter is sufficient if they are "standard," that is, proportioned after either the Sellers (U. S. Standard) or Whitworth (English Standard) systems, as the proportions of heads and nuts, number of threads to the inch, etc., can be obtained from the tables in the Appendix. If not standard note the number of threads to the inch. Record whether a screw is right- or left-handed. If right-handed it will advance if turned clock-wise. The shape of thread, whether triangular or square, would also be noted.

On cog, or "gear," wheels obtain the distance between centres and the number of teeth on each wheel. The remaining data are then obtained by calculation.

In taking bridge notes there would be required general sketches of front and end view; and of the flooring system, showing arrangement of tracks, ties, guard-beam and sidewalk; a cross-section; also detail drawings of the top and foot of each post-connection in one longitudinal line from one end to the middle of the structure. In case of a double-track bridge the outside rows of posts are alike but differ from those of the middle truss.

The foregoing hints might be considerably extended to embrace other and special cases, but experience will prove a sufficient teacher if the student will act on the suggestions given and will remember that to get an excess of data is to err on the side of safety. He should take his notes on as large a scale as possible and so index them that drawings of parts may readily be understood in their relation to the whole.

It need hardly be added that the hints given are intended to be merely a partial summary of the instructions which would be given in the more or less brief practice in technical sketching which, presumably, constitutes a part of every course in Graphics; and that unless the draughtsman can be under the direction of a teacher he will be able to sketch much more intelligently after studying

more of the theory involved in Mechanical Drawing and given in the later pages of this work.

LETTERING.

26. In Figs. 7 and 8 the dimensions are given in numerals belonging to the *Extended Gothic* alphabet below, now growing in favor for this purpose. By giving feet to the 1 and 4, as shown

Fig. 10.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

in those figures, they are no longer strictly Gothic in form yet their appearance is enhanced in the opinion of many.

The *Italic Gothic* and *Italic Roman* (Fig. 11) are also favorite types with the engineering profession for general lettering as well as for dimensioning, and the draughtsman should thoroughly master

Fig. 11.

them as among the most serviceable he can employ. They should, like the Gothic, be invariably drawn free-hand when less than a quarter of an inch in height. Use Miller Bros. "Carbon" pen, No. 4, or any equally flexible fine stub for Gothic forms, and either the "Falcon" or Gillott's No. 303 for Roman and Italic Roman letters.

CONVENTIONAL REPRESENTATIONS.

27. Conventional representations of the natural features of the country or of the materials of construction are so called on the assumption, none too well founded, that the engineering profession

Fig. 12.

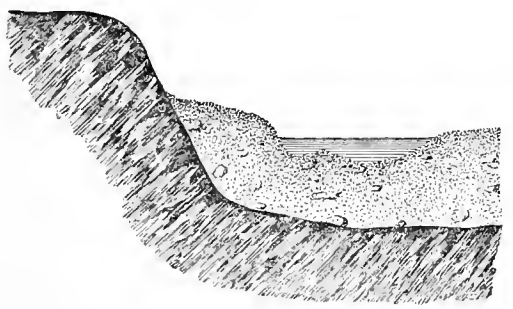
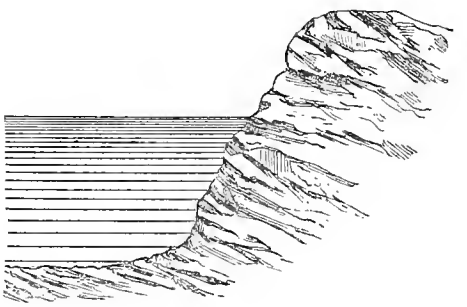


Fig. 13.



has agreed in convention that they shall indicate that which they also more or less resemble. While there is no universal agreement in this matter there is usually but little ambiguity in their use, especially in those that are drawn free-hand, since in them there can be a nearer approach to the natural appearance. This is well illustrated by the sections of river-beds presented in Figs. 12 and

13, the former showing, in addition to the rock cutting, the method of indicating a mud or sand bed with small random boulders. Water either in section or as a receding surface may be shown by parallel lines, the spaces between them increasing gradually.

The conventional representations of wood, masonry and the metals will be found in a succeeding chapter, after hints on coloring have been given, the foregoing figures appearing at this point merely to illustrate the third division of technical free-hand work. Those, however, who have already had some practice in drawing may undertake Figs. 12 and 13 with *burnt umber* undertone for the earthy bed, *pale blue* or *india ink* tint for the rock-section and *prussian blue* for the water-lines.

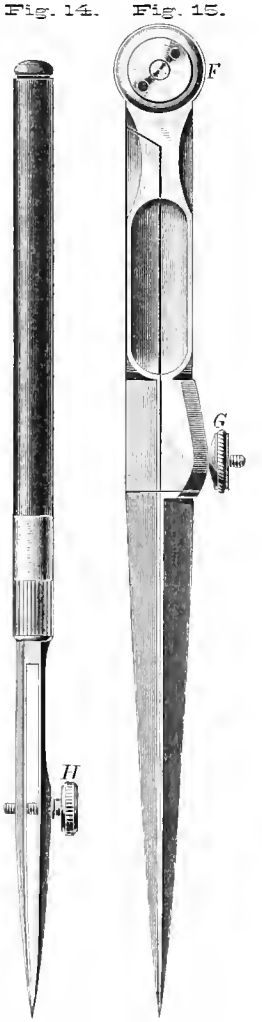
CHAPTER III.

DRAWING INSTRUMENTS AND MATERIALS.—INSTRUCTIONS AS TO USE.—GENERAL PRELIMINARIES AND TECHNICALITIES.

28. The draughtsman's equipment for graphical work should be the best consistent with his means. It is mistaken economy to buy inferior instruments. The best obtainable will be found in the end to have been the cheapest.

The set of instruments illustrated in the following figures contains only those which may be considered absolutely essential for the beginner.

Fig. 14. Fig. 15.



THE DRAWING PEN.

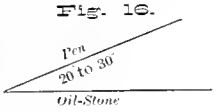
The right line pen (Fig. 14) is ordinarily used for drawing straight lines, with either a rule or triangle to guide it; but it is also employed for the drawing of curves when directed in its motion by curves of wood or hard rubber. For average work a pen about five inches long is best.

The figure illustrates the most approved type, i. e., made from a single piece of steel. The distance between its points, or "nibs," is adjustable by means of the screw *H*. An older form of pen has the outer blade connected with the inner by a hinge. The convenience with which such a pen may be cleaned is more than offset by the certainty that it will not do satisfactory work after the joint has become in the slightest degree loose and inaccurate through wear.

29. If the points wear unequally or become blunt the draughtsman may sharpen them readily himself upon a fine oil-stone. The process is as follows:

Screw up the blades till they nearly touch. Incline the pen at a small angle to the surface of the stone and draw it lightly from left to right (supposing the initial position as in Fig. 16.) Before reaching the right end of the stone begin turning the pen in a plane perpendicular to the surface and draw in the opposite direction at the same angle. After frequent examination and trial to see that the blades have become equal in length and similarly rounded the process is completed by lightly dressing the outside of each blade separately upon the stone. No grinding should be done on the inside of the blade. Any "burr" or rough edge resulting from the operation may be removed with fine emery paper. For the best results, obtained in the shortest possible time, a magnifying glass should be used. The student should take particular notice of the shape of the pen points when new, as a standard to be aimed at when compelled to act on the above suggestions.

Fig. 16.

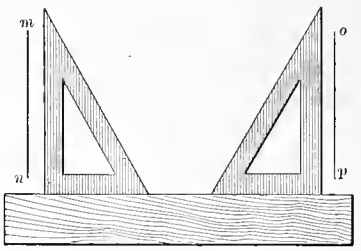


30. The pen may be supplied with ink by means of an ordinary writing pen dipped in the ink and then passed between the blades; or by using in the same manner a strip of Bristol board about a quarter of an inch in width. Should any fresh ink get on the outside of the pen it must

be removed; otherwise it will be transferred to the edge of the rule and thence to the paper, causing a blot.

31. As with the pencil, so with the pen, horizontal lines are to be drawn *from left to right*, while vertical or inclined lines are drawn either from or towards the worker, according to the position of the guiding edge with respect to the line to be drawn. If the line were $m\,n$, Fig. 17, the motion would be away from the draughtsman, i. e., from n toward m; while $o\,p$ would be drawn *toward* the worker, being on the right of the triangle.

Fig. 17.



32. To make a sharply defined, clean-cut line—the only kind allowable—the pen should be held lightly but firmly with one blade resting against the guiding edge, and with both points resting equally upon the paper so that they may wear at the same rate.

33. The inclination of the pen to the paper may best be about 70°. When properly held the pen will make a line about a fortieth of an inch from the edge of the rule or triangle, leaving visible a white line of the paper of that width. If, then, we wish to connect two points by an inked straight line the rule must be so placed that its edge will be from them the distance indicated.

It need hardly be said that a *drawing*-pen should not be *pushed*.

The more frequently the draughtsman will take the trouble to clean out the point of the pen and supply fresh ink the more satisfactory results will he obtain. When through with the pen clean it carefully and lay it away with the points not in contact. Equal care should be taken of all the instruments and for cleaning them nothing is superior to chamois skin.

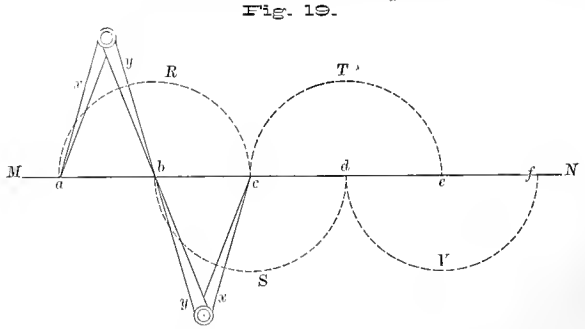
DIVIDERS.

34. The hair-spring dividers (Fig. 15) are employed in dividing lines and spacing off distances, and are capable of the most delicate adjustment by means of the screw G and spring in one of the legs. When but one pair of dividers is purchased the kind illustrated should have the preference over plain dividers, which lack the spring. It will, however, be frequently found convenient to have at hand a pair of each. Should the joint at F become loose through wear it can be tightened by means of a key having two projections which fit into the holes shown in the joint.

35. In spacing off distances the pressure exerted should be the slightest consistent with the location of a point, the puncture to be merely in the surface of the paper and the points determined by lightly pencilled circles about them, thus ——⊙——⊙——. In laying off several equal distances along a line all the arcs described by one leg of the dividers should be on the same side of the line. Thus, in Fig. 19, with b the first centre of turning, the leg x describes the

Fig. 18.

Fig. 19.

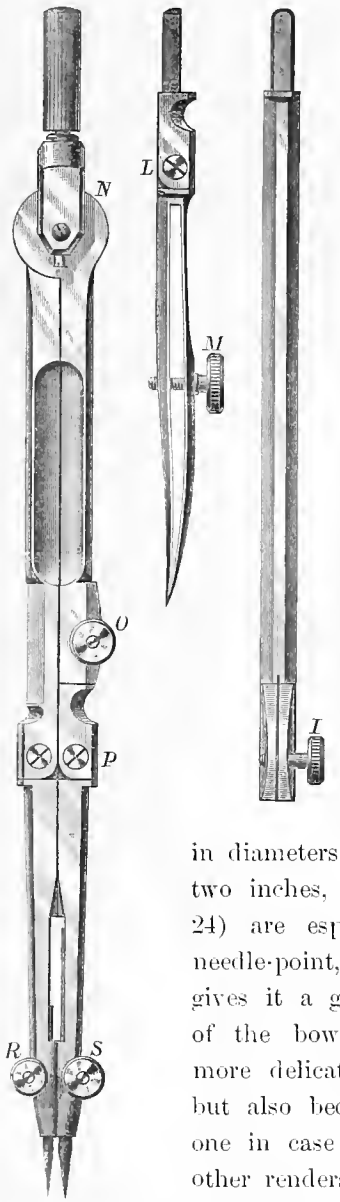


arc R, then rests and pivots on c while the leg y describes the arc S; x then traces arc T, &c.

36. The compasses (Fig. 20) resemble the dividers in form and *may* be used to perform the same office, but are usually employed for the drawing of circles. Unlike the dividers one or both of the legs of compasses are detachable. Those illustrated have one permanent leg, with pivot or "needle-point" adjustable by means of screw *R*. The other leg is detachable by turning the screw *O*, when the pen leg *L M* (Fig. 21) may be inserted for ink work; or the lengthening bar on the right (Fig. 22) may be first attached at *O* and the pencil or pen leg then inserted at *I*, where large work is involved. The metallic point held by screw *S* is usually replaced by a hard lead, sharpened as indicated in Art. 54.

COMPASS SET.

Fig. 20. Fig. 21. Fig. 22.



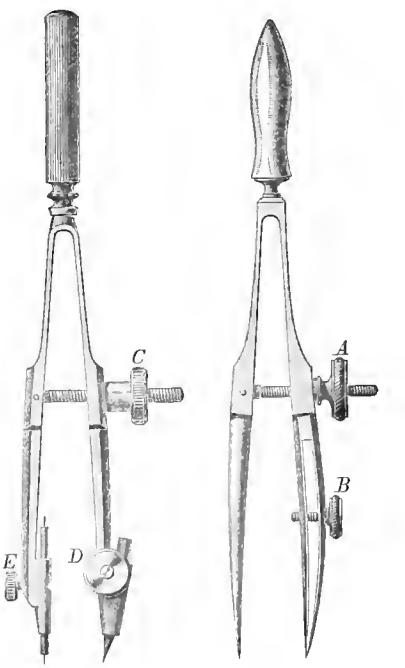
37. When in use the legs should be bent at the joints *P* and *L* so that they will stand perpendicular to the paper if the compasses are held in a vertical plane. The turning may be in either direction but is usually "clock-wise;" and the compasses may be slightly inclined toward the direction of turning. When so used, and if no undue pressure be exerted on the pivot leg, there should be but the slightest puncture at the centre, while the pen points having rested equally upon the paper have sustained equal wear and the resulting line has been sharply defined on both sides. Obviously the legs must be re-adjusted as to angle for any material change in the size of the circles wanted.

The compasses should be held and turned by the milled head, which projects above the joint *N*.

Dividers and compasses should open and shut with an absolutely uniform motion and somewhat stiffly.

BOW-PENCIL AND PEN.

Fig. 23. Fig. 24.



38. For extremely accurate work, in diameters from one-sixteenth of an inch to about two inches, the *bow-pencil* (Fig. 23) and *bow-pen* (Fig. 24) are especially adapted. The pencil-bow has a needle-point, adjustable by means of screw *E*, which gives it a great advantage over the fixed pivot-point of the bow-pen, not alone in that it permits of more delicate adjustment for unusually small work but also because it can be easily replaced by a new one in case of damage; whereas an injury to the other renders the whole instrument useless. For very small circles the needle point should project *very slightly* beyond the pen-point; theoretically by only the extremely small distance the needle-point is expected to sink into the paper.

The spring of either bow should be strong; otherwise an attempt at a circle will result in a spiral.

It will save wear upon the threads of the milled heads *A* and *C* if the draughtsman will press the legs of the bow together with his left hand and run the head up loosely on the screw with his right.

39. To the above described—which we may call the *minimum* set of instruments—might be advantageously added a pair of bow-spacers (small dividers shaped like Fig. 24); beam-compasses, for extra large circles; parallel-rule; proportional dividers and an extra—and larger—right-line pen.

40. The remainder of the *necessary* equipment consists of paper; a drawing-board; T-rule; triangles or "set squares;" scales; pencils; India ink; water colors; saucers for mixing ink or colors; brushes; water-glass and sponge; irregular (or "French") curves; india rubber; erasing knife; protractor; file for sharpening pencils, or a pad of fine emery or sand paper; thumb-tacks (or "drawing pins"); horn centre, for making a large number of concentric circles.

PAPER AND TRACING CLOTH.

41. Drawing paper may be purchased by the sheet or roll and either unmounted or mounted, i. e., "backed" by muslin or heavy card-board. Smooth or "hot-pressed" paper is best for drawings in line-work only; but the rougher surfaced, or "cold-pressed, should always be employed when brush-work in ink or colors is involved: in the latter case also either mounted paper should be used or the sheets "stretched" by the process described in Art. 44.

42. The names and sizes of sheets are:—

Cap 13 × 17	Elephant 23 × 28
Demy 15 × 20	Atlas 26 × 34
Medium 17 × 22	Columbia 23 × 35
Royal 19 × 24	Double Elephant 27 × 40
Super Royal 19 × 27	Antiquarian 31 × 53

Imperial 22 × 30

43. There are many makes of first-class papers, but the best known and still probably the most used is Whatman's. The draughtsman's choice of paper must, however, be determined largely by the value of the drawing to be made upon it and by the probable usage to which it will be subjected.

Where several copies of one drawing were desired it has been a general practice to make the original, or "construction" drawing, with the *pencil*, on paper of a medium grade, then to lay over it a sheet of *tracing-cloth* and copy upon it, in ink, the lines underneath. Upon placing the tracing cloth over a sheet of sensitized paper, exposing both to the light and then immersing the sensitive paper in water a copy or print of the drawing was found upon the sheet, in white lines on a blue ground—the well-known *blue-print*. The time of the draughtsman may, however, be economized as also his purse, by making the original drawing in ink upon Crane's *Bond* paper, which combines in a remarkable degree the qualities of transparency and toughness. About as clear blue-prints can be made with it as with tracing-cloth yet it will stand severe usage in the shop or the drafting-room.

Better papers may yet be manufactured for such purposes and the progressive draughtsman will be on the alert to avail himself of these as of all genuine improvements upon the materials and instruments heretofore employed.

44. To stretch paper tightly upon the board lay the sheet right side up*, place the long rule with its edge about one-half inch back from each edge of the paper in turn and fold up against it a margin of that width. Then *thoroughly* dampen the *back* of the paper with a full sponge, except on the folded margins. Turning the paper again face up, gum the margins with strong mucilage or glue and quickly but firmly press *opposite* edges down simultaneously, long sides first, exerting at the same time a slight outward pressure with the hands to bring the paper down somewhat closer to

*The "right side" of a sheet is, presumably, that towards one when—on holding it up to the light—the manufacturer's name, in water-mark, reads correctly.

the board. Until the gum "sets," so that the paper adheres perfectly where it should, the latter should not shrink; hence the necessity for so completely soaking it at first. The sponge *may* be applied to the *face* of the paper provided it is not *rubbed* over the surface, so as to damage it. When drying, the stretch should be horizontal and no excess of water should be left standing on the surface; otherwise a water-mark will form at the edge of each pool.

45. When tracing-*cloth* is used it must be fastened smoothly, with thumb-tacks, over the drawing to be copied and the ink lining done upon the glazed side, any brush work that may be required —either in ink or colors—being always done upon the dull side of the cloth after the outlining has been completed.

Tracing-cloth, like drawing paper, is most convenient to work upon if perfectly flat. When either has been purchased by the roll it should therefore be cut in sheets and laid away for some time in drawers to become flat before needed for use.

DRAWING BOARD.

46. The drawing board should be slightly larger than the paper for which it is designed and of the most thoroughly seasoned material, preferably some *soft* wood, as pine, to facilitate the use of the drawing-pins or thumb-tacks. To prevent warping it should have battens of hard wood dovetailed into it across the back, transversely to its length. The back of the board should be grooved longitudinally to a depth equal to half the thickness of the wood, which weakens the board transversely and to that degree facilitates the stiffening action of the battens.

It will be found convenient for each student in a technical school to possess two boards, one 20″ × 28″ for paper of Super Royal size, which is suitable for much of a beginner's work, and another 28″ × 41″ for Double Elephant Sheets (about twice Super Royal size) which are well adapted to large drawings of machinery, bridges, &c. A large board may of course be used for small sheets and the expense of getting a second board avoided; but it is often a great convenience to have a medium-sized board, especially in case the student desires to do some work outside the draughting-room.

THE T-RULE.

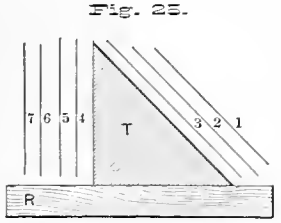
47. The T-rule should be slightly shorter than the drawing board. Its head and blade must have absolutely straight edges and be so rigidly combined as to admit of no lateral play of the latter in the former. The head should also be so fastened to the blade as to be level with the surface of the board. This permits the triangles to slide freely over the head, a great convenience when the lines of the drawing run close to the edge of the paper. (See Fig. 32.)

TRIANGLES.

48. Triangles, or "set-squares" as they are also called, can be obtained in various materials, as hard rubber, celluloid, pear-wood, mahogany and steel; and either solid (Fig. 25) or open (Fig. 26). The open triangles are preferable and two are required, one with acute angles of 30° and 60°, the other with 45° angles. Hard rubber has an advantage over metal or wood, the latter being likely to warp and the former to rust, unless nickel-plated.

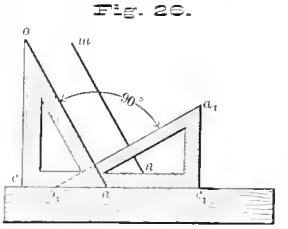
The most frequently recurring problems involving the use of the triangles are the following:—

Fig. 25.



49. *To draw parallel lines* place either of the edges against another triangle or the T-rule. If moved along in either direction each of the other edges will take a series of parallel positions.

Fig. 26.

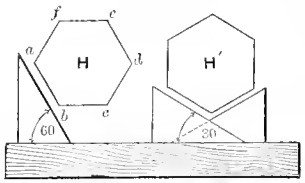


50. *To draw a line perpendicular to a given line* place the *hypothenuse* of the triangle, *o a*, (Fig. 26), so as to coincide with or be parallel to the given line; then a rule or another triangle against the

base. By then turning the triangle so that the other side, $o\,c$, of its right angle shall be against the rule, as at $o_1\,c_1$, the hypothenuse will be found perpendicular to its first position and therefore to the given line.

51. *To construct regular hexagons* place the *shortest* side of the 60° triangle against the rule (Fig. 27) if two sides are to be horizontal, as $f\,e$ and $b\,c$ of hexagon H. For vertical sides, as in H', the position of the triangle is evident. By making $a\,b$ indefinite at first and knowing $b\,c$, the length of a side, we may obtain a by an arc, centre b, radius $b\,c$.

Fig. 27.



If the inscribed circles were given the hexagons might also be obtained by drawing a series of tangents to the circles with the rule and triangles in the positions indicated.

THE SCALE.

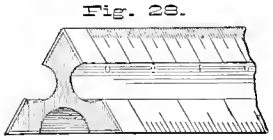
52. But rarely can a drawing be made of the same size as the object, or "full-size," as it is called. The lines of the drawing therefore usually bear a certain ratio to those of the object. This ratio is called the *scale* and should invariably be indicated.

If six inches on the drawing represent one foot on the object the scale is *one-half* and might be variously indicated, thus: SCALE $\frac{1}{2}$; SCALE 1:2; *SCALE 6 IN = 1 FT.* *SCALE 6″ = 1′.*

At one foot to the inch any line of the drawing would be one-twelfth the actual size and the fact indicated *SCALE 1 IN. = 1 FT.*

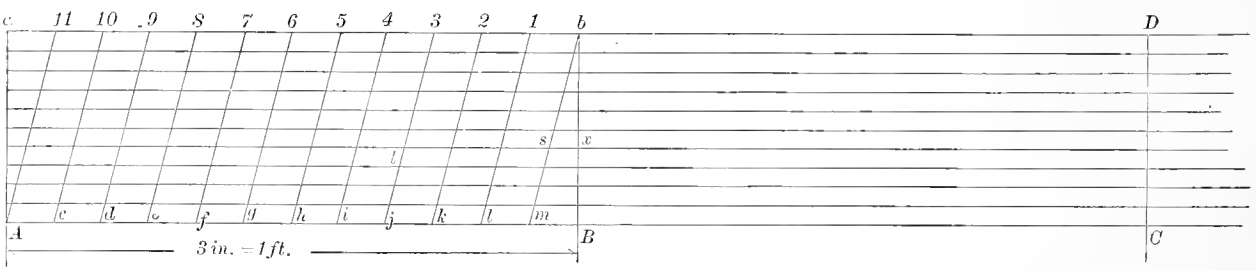
Although it is a simple matter for the draughtsman to make a scale for himself for any particular case yet scales can be purchased in great variety, the most serviceable of which for the usual range of work is of box-wood, 12″ long, (or 18″, if for large work) of the form illustrated by Fig. 28 and graduated $\frac{3}{32}$: $\frac{3}{16}$: $\frac{1}{8}$: $\frac{1}{4}$: $\frac{3}{8}$: $\frac{1}{2}$: $\frac{3}{4}$: 1: $1\frac{1}{2}$: 3 inches to the foot. This is known as the *architect's* scale in contradistinction to the *engineer's*, which is decimally graduated. It will, however, be frequently convenient to have at hand the latter as well as the former.

Fig. 28.



When in use it should be laid along the line to be spaced and a light dot made upon the paper with the pencil, opposite the proper division on the graduated edge. A distance should rarely be transferred from the scale to the drawing by the dividers, as such procedure damages the scale if not the paper.

Fig. 29.



53. For special cases *diagonal* scales can readily be constructed. If, for example, a scale of 3 inches to the foot is needed and measuring to *fortieths of inches*, draw eleven equidistant, parallel lines, enclosing ten equal spaces, as in Fig. 29, and from the end A lay off $A\ B$, $B\ C$, &c., each 3 inches and representing a foot. Then twelve parallel diagonal lines in the first space intercept quarter-inch spaces on $A\ B$ or $a\ b$, each representative of an inch. There being ten equal spaces between B and b, the distance $s\ x$, of the diagonal $b\ m$ from the vertical $b\ B$, taken on any horizontal line $s\ x$, is as many

tenths of the space $m\ B$ as there are spaces between $s\ x$ and b; *six*, in this case. The principle of construction may be generalized as follows:—

The distance apart of the vertical lines represents the units of the scale, whether inches, feet, rods or miles. Except for decimal graduation, divide the left-hand space at top and bottom into as many spaces as there are units of the next lower denomination in one of the original units (feet, for yards as units; inches in case of feet, &c.). Join the points of division by diagonal lines. And if $\frac{1}{x}$ is the smallest fraction that the scale is designed to give rule $x+1$ equidistant horizontal lines, giving x equal horizontal spaces. The scale will then read to $\frac{1}{x}$th of the intermediate denomination of the scale.

When a scale is properly used the spaces on it which represent feet and inches are treated as if they were such in fact. On a scale of one-eighth actual size the edge graduated $1\frac{1}{2}$ inches to the foot would be employed, each $1\frac{1}{2}$ inch space on the scale would be read as if it were a foot; and ten inches, for example, would be ten of the eighth-inch spaces, each of which is to represent an inch of the original line being scaled. The usual error of beginners would be to divide each original dimension by eight and lay off the result, actual size. The former method is by far the more expeditious.

THE PENCILS.

54. For construction lines afterward to be inked the pencils should be of *hard* lead, grade 6H if Faber's or VVH if Dixon's. The pencilling should be *light*. It is easy to make a groove in the paper by exerting too great pressure when using a hard lead. The hexagonal form of pencil is usually indicative of the finest quality and has an advantage over the cylindrical in not rolling off when on a board that is slightly inclined.

Somewhat softer pencils should be used for drawings afterwards to be traced and for the preliminary free-hand sketches from which exact drawings are to be made; also in free-hand lettering.

Sharpen to a *chisel* edge for work along the edges of the T-rule or triangles, but use another pencil with *coned* point for marking off distances with a scale, locating centres and other isolated points and for free-hand lettering; also sharpen the compass leads to a point. Use the knife for cutting the wood of the pencil, beginning at least an inch from the end. Leave the lead exposed for a quarter of an inch and shape it as desired either with a knife or on a fine file or a pad of fine emery paper.

THE INK.

55. Although for many purposes some of the liquid drawing-inks now in the market, particularly Higgins', answer admirably, yet for the best results, either with pen or brush, the draughtsman should mix the ink himself with a stick of India—or, more correctly, *China* ink, selecting one of the higher-priced cakes of rectangular cross-section. The best will show a lustrous, almost iridescent fracture and will have a smooth, as contrasted with a gritty *feel* when tested by rubbing the moistened finger on the end of the cake.

Sets of saucers, called "nests," designed for the mixing of ink and colors, form an essential part of an equipment. There are usually six in a set and so made that each answers as a cover for the one below it. Placing from fifteen to twenty drops of water in one of these the stick of ink should be rubbed on the saucer with *moderate* pressure.

To properly mix ink requires great patience, as with too great pressure a mixture results having flakes and sand-like particles of ink in it, whereas an absolutely smooth and rather thick, slow-flowing liquid is wanted, whose surface will reflect the face like a mirror. The final test as to sufficiency of grinding is to draw a broad line and let it dry. It should then be a rich jet black with a slight

lustre. The end of the cake must be carefully dried on removing it from the saucer, to prevent its flaking, as it will otherwise invariably do.

One may say, almost without qualification, and particularly when for use on tracing-cloth, the thicker the ink the better; but if it should require thinning, on saving it from one day to another—which is possible with the close-fitting saucers described—add a few drops of water, or of ox-gall if for use on a glazed surface.

When the ink has once dried on the saucer no attempt should be made to work it up again into solution. Clean the saucer and start anew.

WATER COLORS.

56. The ordinary colored writing inks should never be used by the draughtsman. They lack the requisite "body" and are corrosive to the pen. Winsor and Newton's water colors, in the form called "moist," and in "half-pans" are the best and most convenient for color work either with pen or brush. Those most frequently employed in engineering and architectural drawing are Prussian Blue, Carmine, Light Red, Burnt Sienna, Burnt Umber, Vermilion, Gamboge, Yellow Ochre, Chrome Yellow, Payne's Gray and Sepia.

Although hardly properly called a color, Chinese White may be mentioned at this point as a requisite, and obtainable of the same form and make as the colors above.

DRAWING-PINS.

57. Drawing-pins or thumb-tacks, for fastening paper upon the board, are of various grades, the best and at present the cheapest being made from a single disc of metal one-half inch in diameter, from which a section is partially cut, then bent at right angles to the surface, forming the point of the pin.

IRREGULAR CURVES.

58. Irregular or French curves, also called *sweeps*, for drawing non-circular arcs, are of great variety and the draughtsman can hardly have too many of them. They may be either of pear wood or hard rubber. A thoroughly equipped draughting office will have a large stock of these curves, which may be obtained in sets, and are known as railroad curves, ship curves, spirals, ellipses, hyperbolas, parabolas and combination curves.

If but two are obtained (which would be a minimum stock for a beginner) the forms shown in Fig. 30 will probably prove as serviceable as any. When employing them for inked work the pen should be so turned, as it advances, that its blades will maintain the same relation (parallelism) to the edge of the guiding curve as they ordinarily do to the edge of the rule. And the student must content himself with drawing slightly less of the curve than might apparently be made with one setting of the sweep, such course being safer in order to avoid too close an approximation to angles in what should be a smooth curve. For the same reason, when placed in a new position, a portion of the irregular curve must coincide with a part of that last inked.

Fig. 30.

The pencilled curve is usually drawn free-hand after a number of the points through which it should pass have been definitely located. In sketching a curve free-hand it is much more naturally and smoothly done if the curve is concave toward the hand.

INDIA RUBBER.

59. For erasing pencil lines and cleaning the paper india rubber is required, that known as "velvet" being recommended for the former purpose, and either "natural" or "sponge" rubber for the latter. Stale bread crumbs are equally good for cleaning the surface of the paper after the lines have been inked, but will damage pencilling to some extent.

One end of the velvet rubber may well be wedge-shaped for erasing lines without damaging others near them.

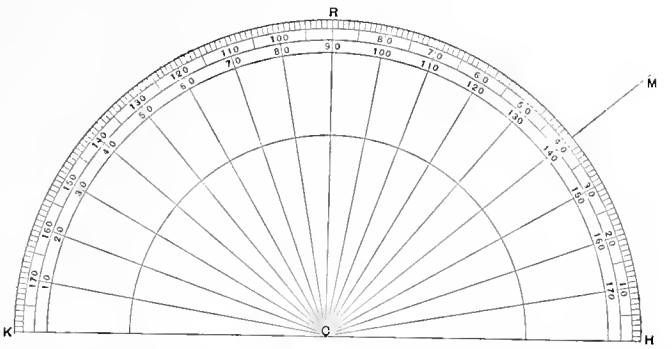
INK ERASER.

60. The double-edged erasing knife gives the quickest and best results when an inked line is to be removed. The point should rarely be employed. The use of the knife will damage the paper more or less, to partially obviate which rub the surface with the thumb nail or an ivory knife handle.

PROTRACTOR.

61. For laying out angles a graduated arc called a "protractor" is employed.

Fig. 31.



These are made of various materials, as metal, horn, celluloid, Bristol board and tracing paper. The two last are quite accurate enough for ordinary purposes, though where the utmost precision is required one of German silver should be obtained, having a movable arm and vernier attachment.

The graduation may advantageously be to half degrees for average work.

To lay out an angle (say 40°) with the protractor, the radius, *C H*, would be made to coincide with one side of the desired angle; the centre *C*, with the desired vertex; and a dot made with the pencil opposite division numbered 40 on the graduated edge. The line, *M C*, through this point and *C*, would complete the construction.

BRUSHES.

62. Sable-hair brushes are the best for laying flat or graduated tints, with ink or colors, upon small surfaces; while those of camel's hair, large, with a brush at each end of the handle, are as well adapted for tinting large surfaces. Reject any brush that does not come to a perfect point on being moistened. Five or six brushes of different sizes are needed.

PRELIMINARIES TO PRACTICAL WORK.

63. The first work of a draughtsman, like most of his later productions, consists of *line* as distinguished from *brush* work, and for it the paper may be fastened upon the board with thumb-tacks only.

There is no universal standard as to *size* of sheets for drawings. As a rule each draughting office has its own set of standard sizes, and system of preserving and indexing. The columns of the various engineering papers present frequent notes on these points, and apparently the best system of preserving and recording drawings, tracings and corrections is in process of evolution. For the student the best plan is to have all drawings of the same size bound in neat but permanent form at the end of the course. The title-pages, which presumably have also been drawn, will sufficiently distinguish the different sets.

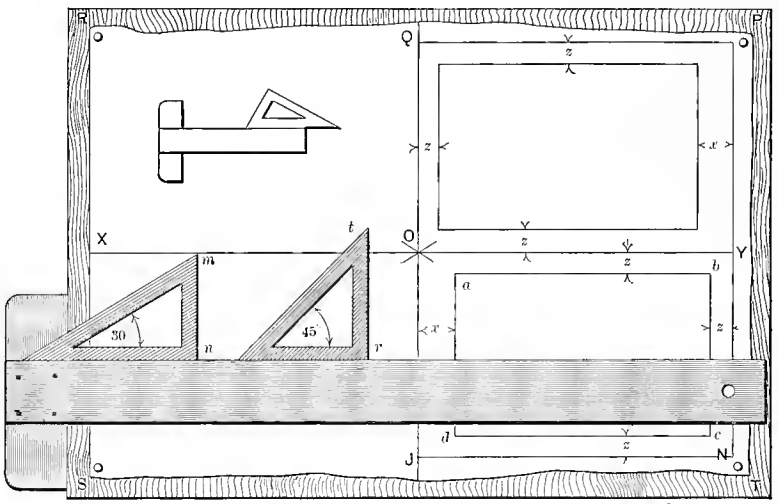
In his elementary work the student may to advantage adopt two sizes of sheets which are considerably employed, $9'' \times 13''$ and its double $13'' \times 18''$; sizes into which a "Super Royal" sheet naturally divides, leaving ample margins for the mucilage in case a "stretch" is to be made.

A "Double Elephant" sheet being twice the size of a "Super Royal" divides equally well into plates of the above size, but is preferable on account of its better quality.

To lay out the rectangles upon the paper, first locate the centre (see Fig. 32) by intersecting diagonals, as at *O*. These should *not* be drawn entirely across the sheet, but one of them will necessarily pass a *short* distance each side of the point where the centre lies—judging by the eye alone; the second definitely determines the point. If the T-rule will not reach diagonally from corner to corner of the paper (and it usually will not) the edge may be practically extended by placing a triangle against but *projecting beyond* it, as in the upper left-hand portion of the figure.

64. The T-rule being placed as shown, with its head *at the left end* of the board—the correct and usual position—draw a horizontal line *X Y*, through the centre just located. The vertical centre line is then to be drawn, with one of the triangles placed as shown in the figure, i. e., so that a side, as *m n* or *t r*, is perpendicular to the edge.

Fig. 32.



It is true that as long as the edges of the board are exactly at right angles with each other we might use the T-rule altogether for drawing mutually perpendicular lines. This condition being, however, rarely realized for any length of time, it has become the custom, a safe one—so long as rule and triangle remain "true," to use them as stated.

The outer rectangles for the drawings (or "plates," in the language of the technical school) are completed by drawing parallels, as *J N* and *Y N*, to the centre lines, at distances from them of 9″ and 13″ respectively, laid off *from the centre, O.*

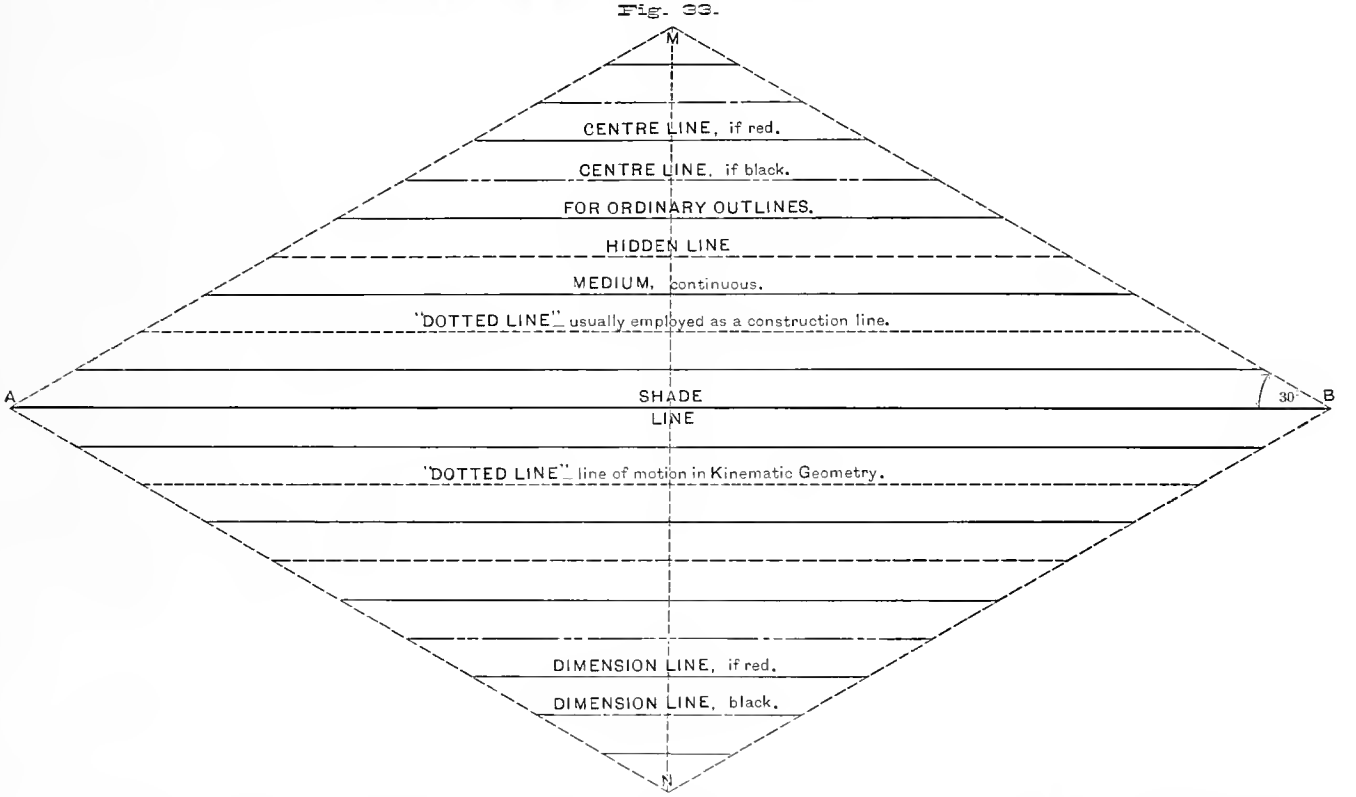
An inner rectangle, as *a b c d*, should be laid out on each plate, with proper margins; usually at least an inch on the top, right and bottom, and an *extra half inch on the left* as an allowance for binding. These margins are indicated by *x* and *z* in the figure, as variables to which any convenient values may be assigned. The broad margin *x* in the upper rectangle will be at the draughtsman's left hand if he turns the board entirely around—as would be natural and convenient—when ready to draw on the rectangle *Q Y*.

CHAPTER IV.

GRADES OF LINES.—LINE TINTING.—LINE SHADING.—CONVENTIONAL SECTION-LINING.—FREQUENTLY RECURRING PLANE PROBLEMS.—MISCELLANEOUS PEN AND COMPASS EXERCISES.

65. The various kinds of lines employed in mechanical drawing are indicated in the figure below, and while getting his elementary practice with the ruling-pen the student may group them as shown or in any other symmetrical arrangement, either original with himself or suggested by other designs.

Fig. 33.



When drawing on tracing cloth or tracing paper, for the purpose of making blue-prints, all the lines will preferably be *black*, and the centre and dimension lines distinguished from others by the alternation indicated, as also by being somewhat finer than those employed for the light outlines of the object drawn. Heavy, opaque, *red* lines may, however, be used, and will blue-print as faint white lines. There is at present no universal agreement among the members of the engineering profession as to standard dimension and centre lines. Not wishing to add another to the systems already at variance, but preferring to facilitate the securing of the uniformity so desirable, I have presented those for some time employed by the Pennsylvania Railroad and now taught at Cornell. The so-called "dotted" line is actually composed of short dashes. Its use, as a "line of motion," was suggested at Cornell. When colors are used without intent to blue-print they may be drawn

as light, continuous lines. Colors will further add to the intelligibility of a drawing if employed for *construction* lines. Even if red dimension lines are used the *arrow heads* should invariably be ***black.*** They should be drawn *free-hand*, with a writing pen, and their points *touch* the lines between which they give the distance.

66. The utmost accuracy is requisite in pencilling, as the draughtsman should be merely a copyist when using the pen. On a complicated drawing even the *kind* of line should be indicated at the outset, so that no time will be wasted, when inking, in the making of distinctions to which thought has already been given during the process of construction. No unnecessary lines should be drawn; or exceeding of the intended limit of a line if it can be avoided.

If the work is symmetrical, in whole or in part, draw centre lines first, then main outlines; and continue the work from large parts to small.

The visible lines of an object are to be drawn first; afterward those indicated as concealed.

All lines of the same quality may to advantage be drawn with one setting of the pen, to ensure uniformity; and the light outlines before the shade lines.

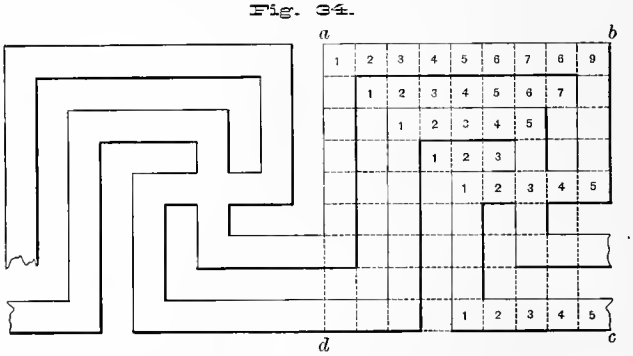
In drawing arcs and their tangents ink the former first, invariably.

All the inking may best be done at once, although for the sake of clearness, in making a large and complicated drawing, a portion—usually the nearest and visible parts—may be inked, the drawing cleaned and the pencilling of the construction lines of the remainder continued from that point.

The inking of the centre, dimension and construction lines naturally follows the completion of the main design.

67. In Fig. 34 we have a straight-line design usually called the "Greek Fret," and giving the student his first illustration of the use of the "shade line" to bring a drawing out "in relief." The law of the construction will be evident on examination of the numbered squares.

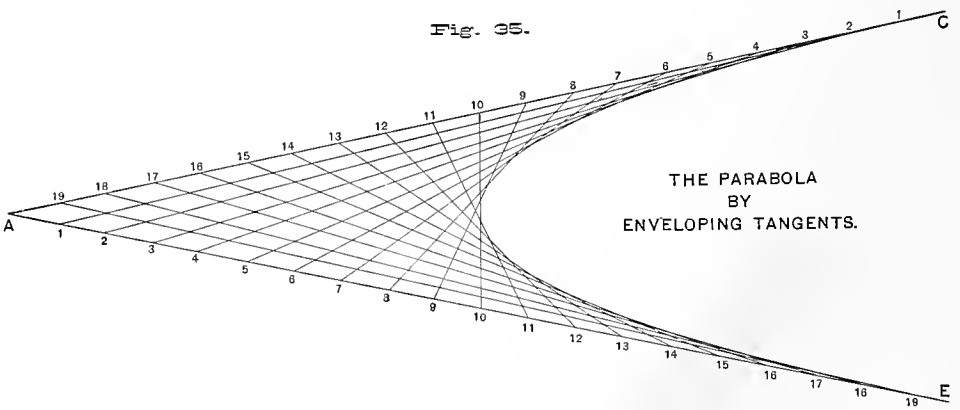
Fig. 34.



Without entering into the theory of shadows at this point we may state briefly the "shop rule" for drawing shade lines, viz., *right-hand and lower*. That is, of any pair of lines making the same turns together or representing the limit of the same flat surface, the *right-hand* line is the heavier if the pair is vertical, but the *lower* if they run horizontally; always subject, however, to the proviso that the line of intersection of two illuminated planes is never a shade line.

68. Another interesting exercise in ruled lines is the curve called the *parabola*, represented by its tangents (Fig. 35). The angle CAE may be assumed at pleasure, and on the finished drawing

Fig. 35.

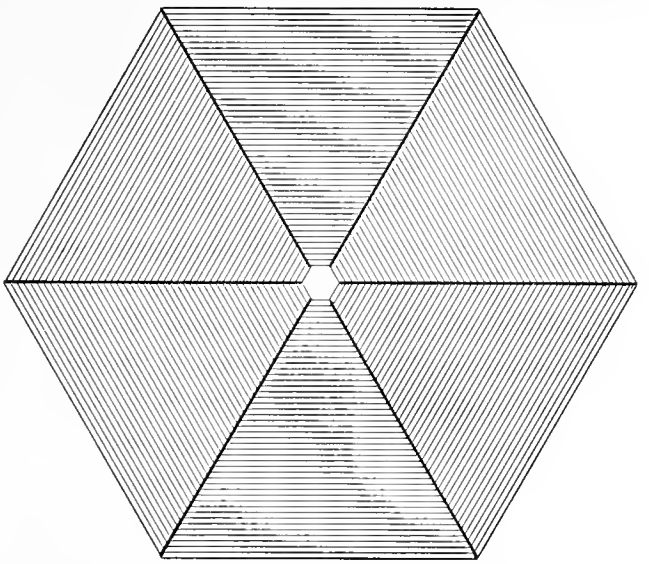


the numbers may be omitted, being given merely to show the law of construction. Like numbers are joined and all spaces equal.

Some interesting mathematical properties of the curve will be found in a subsequent chapter.

69. A pleasing design that will test the beginner's skill is that of Fig. 36. It is suggestive of a cobweb; and a skillful free-hand draughtsman could make it more realistic by adding the spider.

Fig. 36.



The 60° triangle may be used for the heavy diagonals and parallels to them, the T-rule giving the horizontals.

70. The even or flat effect of equidistant parallel lines is called *line-tinting;* or, if representing an object that has been cut by a plane, as in Fig. 37, it is called *section-lining.*

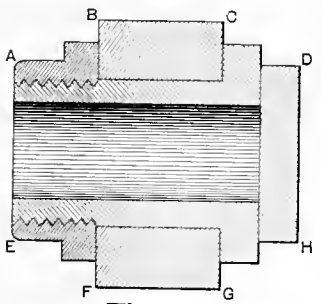


Fig. 37.

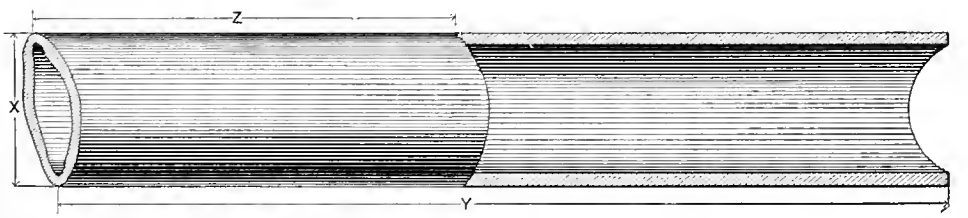
The *section*, strictly speaking, is the part actually in contact with the cutting plane; while the drawing as a whole is a *sectional view*, as it also shows what is back of the plane of section—the latter being, presumably, transparent.

Adjacent pieces have the lines drawn in different directions in order to distinguish sufficiently between them.

The curved effect on the semi-cylinder is evidently obtained by properly varying both the strength of the lines and the spacing.

71. The difference between the shading on the exterior and interior of a cylinder is sharply contrasted in Fig. 38. The two right-hand curves might preferably be more elliptical, and *tangent* to the straight lines on which they terminate.

Fig. 38.



The spacing of the lines, in section-lining, depends upon the scale of the drawing. It may run down to a thirtieth of an inch or as high as one-eighth; but from a twentieth to a twelfth of an

inch would be best adapted to the ordinary range of work. *Equal* spacing and not fine spacing should be the object, and neither scale nor patent section-liner should be employed, but distances gauged by the eye alone.

72. A refinement in execution which adds considerably to the effect is to leave a white line between the top and left-hand outlines of each piece and the section lines. When purposing to produce this effect rule light pencil lines as limits for the line-tints.

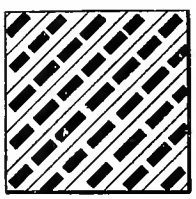
73. If the various pieces shown in a section are of different materials there are four ways of denoting the difference between them:—

(*a.*) By the use of the brush and certain water-colors, a method considerably employed in Europe, but not used to any great extent in this country, probably owing to the fact that it is not applicable where blue-prints of the original are desired.

The use of colors may, however, be advantageously adopted when making a highly finished, shaded drawing; the shading being done first, in india ink or sepia, and then overlaid with a flat tint of the conventional color. The colors ordinarily used for the metals are

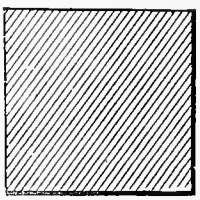
Payne's gray or India ink	for	Cast Iron.
Gamboge	"	Brass (outside view).
Carmine	"	Brass (in section).
Prussian Blue	"	Wrought Iron.
Prussian Blue with a tinge of Carmine	"	Steel.

Fig. 39.

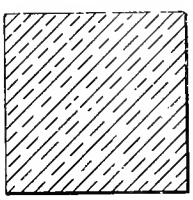


Cast Iron.

Steel.



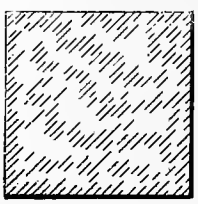
Wr't. Iron.



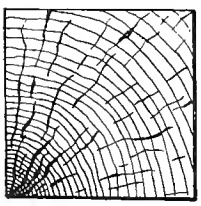
Brass.

PENNA. R. R.

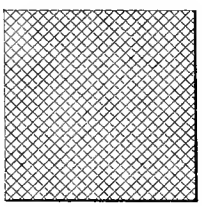
Standard Sections



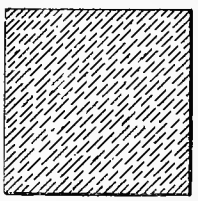
Stone.



Wood.



Copper.



Brick.

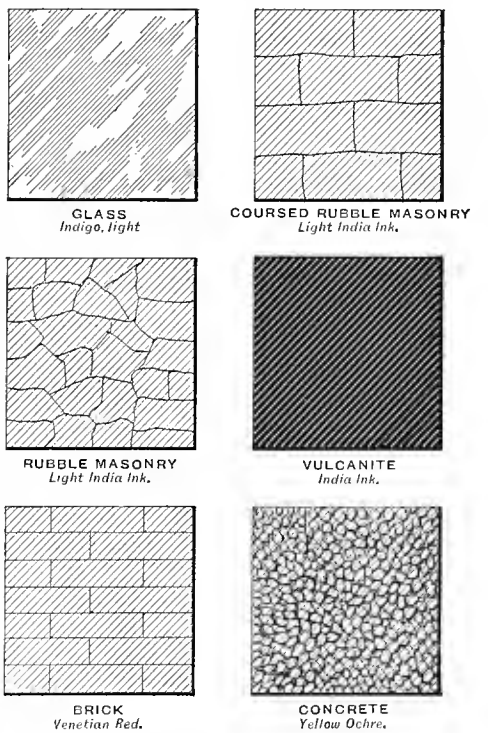
More natural effects can also be given by the use of colors, in representing the other materials of construction; and the more of an artist the draughtsman proves to be the closer can he approximate to nature.

Pale blue may be used for *water lines;* Burnt Sienna, whether grained or not, suggests *wood;* Burnt Umber is ordinarily employed for *earth;* either Light Red or Venetian Red are well adapted for *brick*, and a wash of India ink having a tinge of blue gives a fair suggestion of *masonry;* although the actual tint and surface of any rock can be exactly represented after a little practice with the brush and colors. These points will be enlarged upon later.

(*b.*) By section-lining with the drawing pen in the conventional colors just mentioned, a process giving very handsome and thoroughly intelligible results on the original drawing but, as before, unadapted to blue-printing and therefore not as often used as either of the following methods.

(*c.*) By section-lining uniformly in *ink* throughout and printing the name of the material upon each piece.

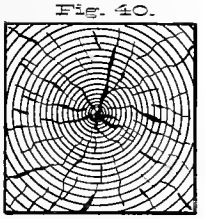
Fig. 41.



(*d.*) By alternating light and heavy, continuous and broken lines according to some law. Said "law" is, unfortunately, by no means universal, despite the attempt made at a recent convention of the American Society of Mechanical Engineers to secure uniformity. Each draughting office seems at present to be a law unto itself in this matter.

74. As affording valuable examples for further exercise with the ruling pen the system of section-lines adopted by the Pennsylvania Railroad is presented on the opposite page. The *wood* section is an exception to the rule, being drawn free-hand with a Falcon pen.

Fig. 40.



By way of contrasting free-hand with mechanical work Fig. 40 is introduced, in which the rings showing annual growth are drawn as concentric circles with the compass.

In Fig. 41 a few other sections appear, selected from the designs of M. N. Forney and F. Van Vleck, and which are fortunate arrangements.

75. Figs. 42 and 43 are profiles or outlines of mouldings, such as are of frequent occurrence in architectural work. It is good practice to convert such views into oblique projections, giving the effect of solidity; and to further bring out their form by line shading. Figs. 44–46 are such representations, the front of each being of the same *form* as Fig. 42. The oblique lines are all parallel to each other and—where

Fig. 42.

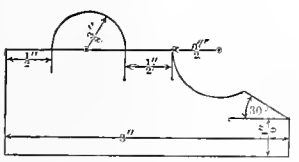


Fig. 43.

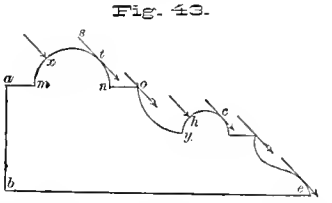
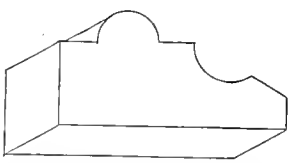


Fig. 44.



visible throughout—of the same length. Their *direction* should be chosen with reference to best exhibiting the peculiar features of the object. Obviously the view in Fig. 44 is the least adapted to the conveying of a clear idea of the moulding, while that of Fig. 46 is evidently the best.

76. The student may, to advantage, design profiles for mouldings and line-shade them, after converting them into oblique views. As hints for such work two figures are given (47–48), taken

Fig. 45.

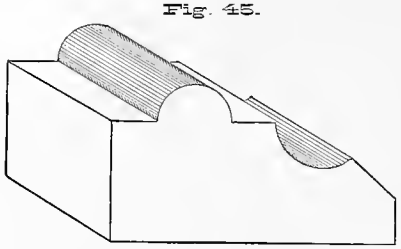
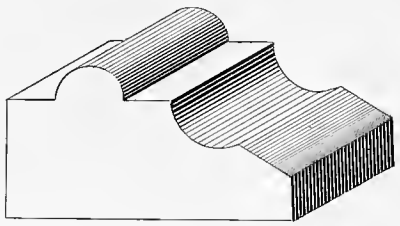


Fig. 46.



from actual constructions in wood. By setting a moulding vertically, as in Fig. 49, and projecting horizontally from its points, a front view is obtained, as in Fig. 50.

Fig. 47.

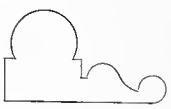


Fig. 48.

Fig. 49.

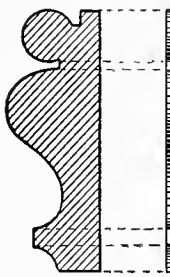
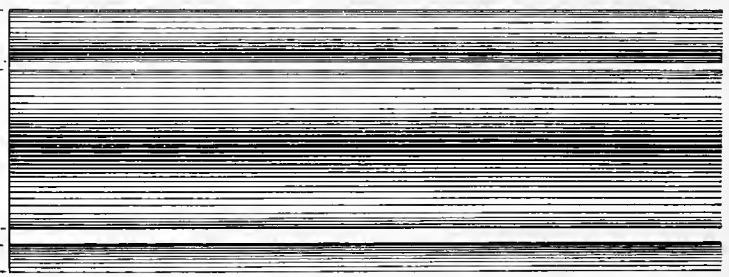
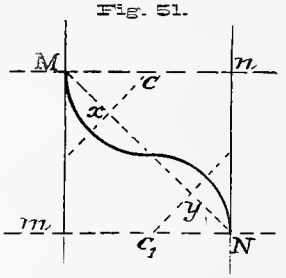


Fig. 50.



77. The *reverse curves* on the mouldings may be drawn with the irregular curve, (see Art. 58); or, if composed of circular arcs to be tangent to vertical lines, by the following construction:—

Fig. 51.



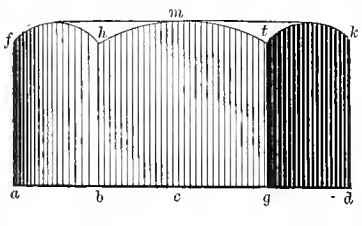
Let M and N be the points of tangency on the verticals $M m$ and $N n$; and let the arcs be tangent to each other at the middle point of the line $M N$. Draw $M n$ and $N m$ perpendicular to the vertical lines. The centres, c and c_1, of the desired arcs are at the intersection of $M n$ and $N m$ by perpendiculars to $M N$ from x and y, the middle points of the segments of $M N$.

78. The light is to be assumed as coming in the usual direction, i. e., descending from left to right at such an angle that any ray would be projected on the paper at an angle of 45° to the horizontal.

In Fig. 43 several rays are shown. At x, where the light strikes the cylindrical portion most directly—technically is *normal* to the surface—is actually the brightest part. A tangent ray $s t$ gives t, the darkest part of the cylinder. The concave portion beginning at o would be darkest at o and get lighter as it approaches y.

Flat parts are to be left either white, if in the light, or have equidistant lines if in the shade; unless the most elegant finish is desired, in which case both change of space and gradation of line must be resorted to as in Fig. 52, which represents a front view of a hexagonal nut. The front face being parallel to the paper receives an even tint. An inclined face *in the light*, as $a\ b\ h\ f$, is lightest toward the observer, while the unillumined face $t\ k\ d\ g$ is exactly the reverse.

Fig. 52.



Notice that to give a *flat* effect on the inclined faces the spacing-out as also the change in the size of lines must be more gradual than when indicating curvature. (Compare with Figs. 46 and 50.)

If two or more illuminated flat surfaces are parallel to the paper (as $t\ g\ b\ h$, Fig. 52) but at different distances from the eye, the nearest is to be the lightest; if unilluminated, the reverse would be the case.

79. In treating of the theory of shadows distinctions have to be made, not necessary here, between *real* and *apparent* brilliant points and lines. We may also remark at this point that to an experienced draughtsman some license is always accorded, and that he can not be expected to adhere rigidly to theory when it involves a sacrifice of effect. For example, in Fig. 46 we are unable to see to the left of the, theoretically, lightest part of the cylinder, and find it, therefore, advisable to move the darkest part past the point where, according to Fig. 43, we know it in reality to be. The professional draughtsmen who draw for the best scientific papers and to illustrate the circulars of the leading machine designers allow themselves the latitude mentioned, with most pleasing results. Yet until one may be justly called an expert he can depart but little from the narrow confines of theory without being in danger of producing decidedly peculiar effects.

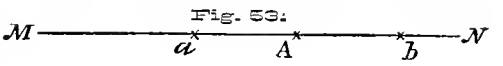
80. As from this point the student will make considerable use of the compasses, a few of the more important and frequently recurring plane problems, nearly all of which involve their use, may well be introduced. The proofs of the geometrical constructions are in several cases omitted, but if desired the student can readily obtain them by reference to any synthetic geometry or work on plane problems.

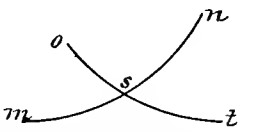
All the problems given (except No. 20) have proved of value in shop practice and architectural work.

The student should again read Arts. 48–51 regarding special uses of the 30° and 45° triangles, which, with the T-rule, enable him to employ so many "draughtsman's" as distinguished from "geometrician's" methods; also Arts. 36 and 37.

81. *Prob. 1. To draw a perpendicular to a given line at a given point*, as A, (Fig. 53), use the triangles, or triangle and rule as previously described; or lay off equal distances $A\ a$, $A\ b$, and with a and b as centres draw arcs $o\ s\ t$, $m\ s\ n$, with common radius greater than one-half $a\ b$. The required perpendicular is the line joining A with the intersection of these arcs.

Fig. 53.

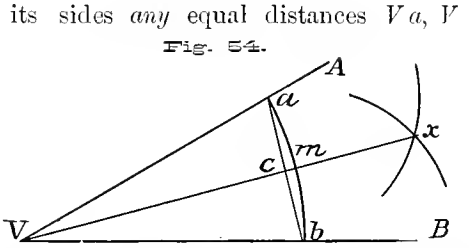




82. *Prob. 2. To bisect a line*, as $M\ N$, use its *extremities* exactly as a and b were employed in the preceding construction, getting also a second pair of arcs (same radius for all the arcs) intersecting *above* the line at a point we may call x. The line from s to x will be a bisecting perpendicular.

83. *Prob. 3. To bisect an angle*, as $A\ V\ B$, (Fig. 54) take on its sides *any* equal distances $V\ a$, $V\ b$. Use a and b as centres for intersecting arcs having a common radius. Join V with x, the intersection of these arcs, for the bisector required.

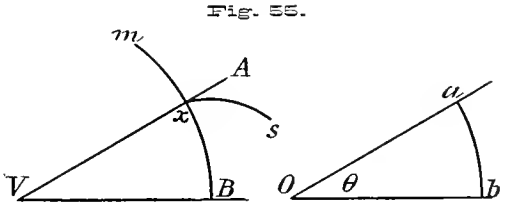
Fig. 54.



84. *Prob. 4. To bisect an arc of a circle*, as $a\ m\ b$ (Fig. 54), bisect the chord $a\ c\ b$ by Prob. 2; or, by Prob. 3, bisect the angle $a\ V\ b$ which subtends the arc.

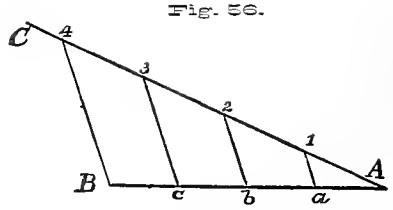
85. *Prob. 5. To construct an angle equal to a given angle*, let θ be the given angle, (Fig. 55) draw *any* arc $a\ b$ with centre O, then, with same radius, an indefinite arc $m\ B$, centre V. Use *chord* of $a\ b$ as a radius and from centre B cut the arc $m\ B$ at x. Join V and x.

Fig. 55.



86. *Prob. 6. To pass a circle through three points* a, b and c, join them by lines $a\ b$, $b\ c$, bisect these lines by perpendiculars and the intersection of the latter will be the centre of the desired circle.

Fig. 56.



87. *Prob. 7. To divide a line into any number of equal parts* draw from one extremity as A (Fig. 56) a line $A\,C$ making any random angle with the given line $A\,B$. With a scale point off on $A\,C$ as many equal parts (size immaterial) as are required on $A\,B$; four, for example. Join the last point of division (4) with B and parallels to such line from the other points will divide $A\,B$ similarly.

88. A *secant* to a curve is a line cutting it in two points. If the secant $A\,B$ be turned to the left about A the point B will approach A and the line at one time coincide with $A\,C$. If turned so that B reaches and coincides with A the line will then touch the curve in but one point and is then called a *tangent*.

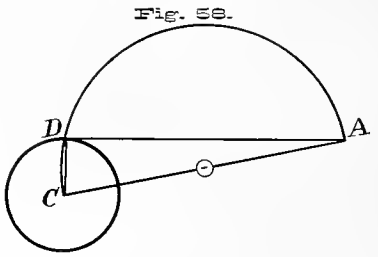
Fig. 57.

The *normal* to a curve is perpendicular to the tangent, at the point of tangency. In a circle it coincides in direction with the radius to the point of tangency.

89. *Prob. 8. To draw a tangent to a circle at a given point* draw a radius to the point. The perpendicular to this radius at its extremity will be the required tangent. Solve with triangles.

90. *Prob. 9. To draw a tangent to a circle from a point without*, join the centre C (Fig. 58) with the given point A; describe a semi-circle on $A\,C$ as a diameter and join A with D, the intersection of the arcs. $A\,D\,C$ equals 90°, being inscribed in a semi-circle; $A\,D$ is therefore perpendicular to radius $C\,D$ at its extremity, hence tangent to given circle.

Fig. 58.

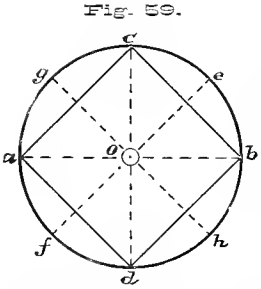


91. *Prob. 10. To draw a tangent at a given point of a circular arc whose centre is unknown or inaccessible*, locate *on the arc* two points equidistant from the given point and on opposite sides of it; join them by a chord and draw through the given point a parallel to the latter.

92. A *regular* polygon has all its sides equal, as also its angles. If of three sides it is called the *equilateral triangle;* four sides, the *square;* five, *pentagon;* six, *hexagon;* seven, *heptagon;* eight, *octagon;* nine, *nonagon* or *enneagon;* ten, *decagon;* eleven, *undecagon;* twelve, *dodecagon.* The angles of the more important regular polygons are

120°	*at the centre*			Triangle	60°	*at the circumference.*	
90°	"	"	"	Square	90°	"	"
72°	"	"	"	Pentagon	108°	"	"
60°	"	"	"	Hexagon	120°	"	"
45°	"	"	"	Octagon	135°	"	"
36°	"	"	"	Decagon	144°	"	"
30°	"	"	"	Dodecagon	150°	"	"

Fig. 59.

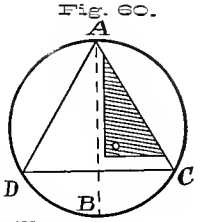


93. For the polygons most frequently occurring there are many special methods of construction. All but the pentagon and decagon can be readily inscribed or circumscribed about a circle by the use of the T-rule and triangles. For example, draw $a\,b$ (Fig. 59) with the T-rule and $c\,d$ perpendicular to it with a triangle. The 45° triangle will then give the square $a\,c\,b\,d$. The same triangle in two positions would give $e\,f$ and $g\,h$, whence $a\,g$, $g\,c$, &c., would be sides of a regular *octagon.*

94. The 60° triangle used as in Art. 51 would give the *hexagon;* and alternate vertices of the latter, joined, would give the equilateral triangle. Or the radius of the circle stepped off six times on the circumference and alternate points connected would result similarly.

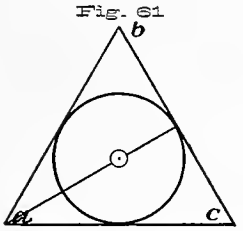
95. *Prob. 11.* An additional method for *inscribing an equilateral triangle in a circle, when one vertex of the triangle is given*, as A, is to draw the diameter, $A\,B$, through A, and use the triangle to obtain the sides $A\,C$ and $A\,D$, making angles of 30° with $A\,B$. D and C will then be the extremities of the third side of the triangle sought.

Fig. 60.



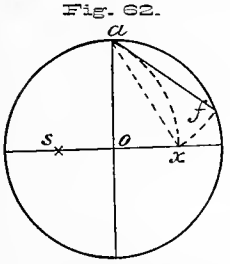
96. *Prob. 12. To inscribe a circle in an equilateral triangle* draw a perpendicular from any vertex to the opposite side. The centre of the circle will be on such line, two-thirds of the distance from vertex to base, while the radius desired will be the remaining third.

Fig. 61



97. *Prob. 13. To inscribe a circle in any triangle* bisect any two of the interior angles. The intersection of these bisectors will be the *centre* and its perpendicular distance from any side the *radius* of the circle sought.

Fig. 62.

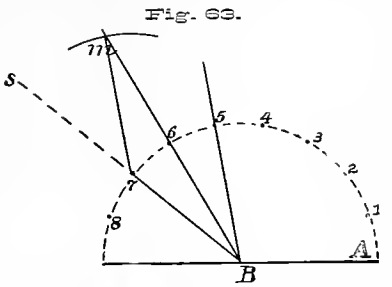


98. *Prob. 14. To inscribe a pentagon in a circle* draw mutually perpendicular diameters (Fig. 62); bisect a radius as at s; draw arc $a\,x$ of radius $s\,a$ and centre s; then chord $a\,x = a\,f$, the side of the pentagon to be constructed.

99. *Prob. 15. To construct a regular polygon of any number of sides, length of side given.*

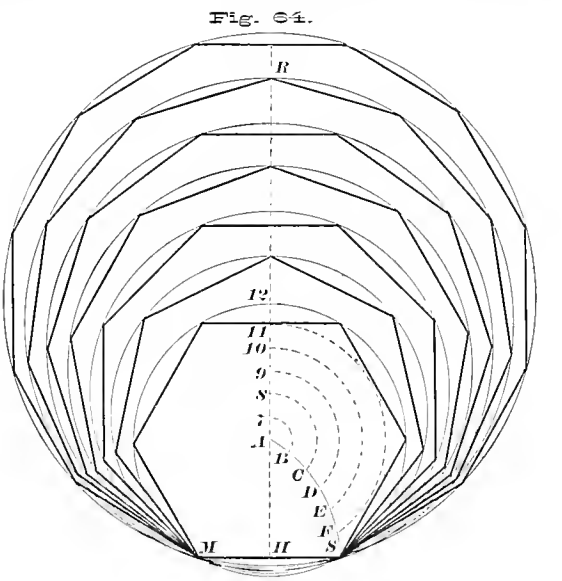
Let $A\,B$ (Fig. 63) be the length assigned to a side, and a regular polygon of x sides desired. Take x equal to *nine* for illustration, draw a semi-circle with $A\,B$ as radius and divide it by trial into x (or 9) equal parts. Join B with x—2 points of division, or *seven*, beginning at A, and prolong all but the last. With 7 as a centre, radius $A\,B$, cut line B 6 at m by an arc and join m with 7, giving another side of the required polygon. Using m in turn as a centre, same radius as before, cut B 5 (produced) and so obtain a third vertex.

Fig. 63.



This solution is based on the familiar principles (a) that if a regular polygon has x sides each interior angle equals $\frac{180°\ (x-2)}{x}$, and (b) that the diagonals drawn from any vertex of the polygon make the same angles with each other as with the sides meeting at that vertex. For $x = 9$ we have $A\ B\ s = \frac{180°\ (9-2)}{9} = \frac{7}{9}$ths of the two right angles about B.

Fig. 64.



100. *Prob. 16. Another solution of Prob. 15.* Erect a perpendicular $H\,R$ (Fig. 64) at the middle point of the given side. With M as a centre, radius $M\,S$, describe arc $S\,A$ and divide it by trial into six equal parts. Arcs through these points of division, using A as a centre, and numbered up from six, give the centres on the vertical line for circles passing through M and S and in which $M\,S$ would be a chord as many times as the number of the centre.

101. For any unusual number of sides the method by "trial and error" is often resorted to, and even for ordinary cases it is by no means to be despised. By it the dividers are set "by guess" to the probable chord of the desired arc and, supposing a *heptagon* wanted, the chord is stepped off seven times around the cir-

PLATE I.

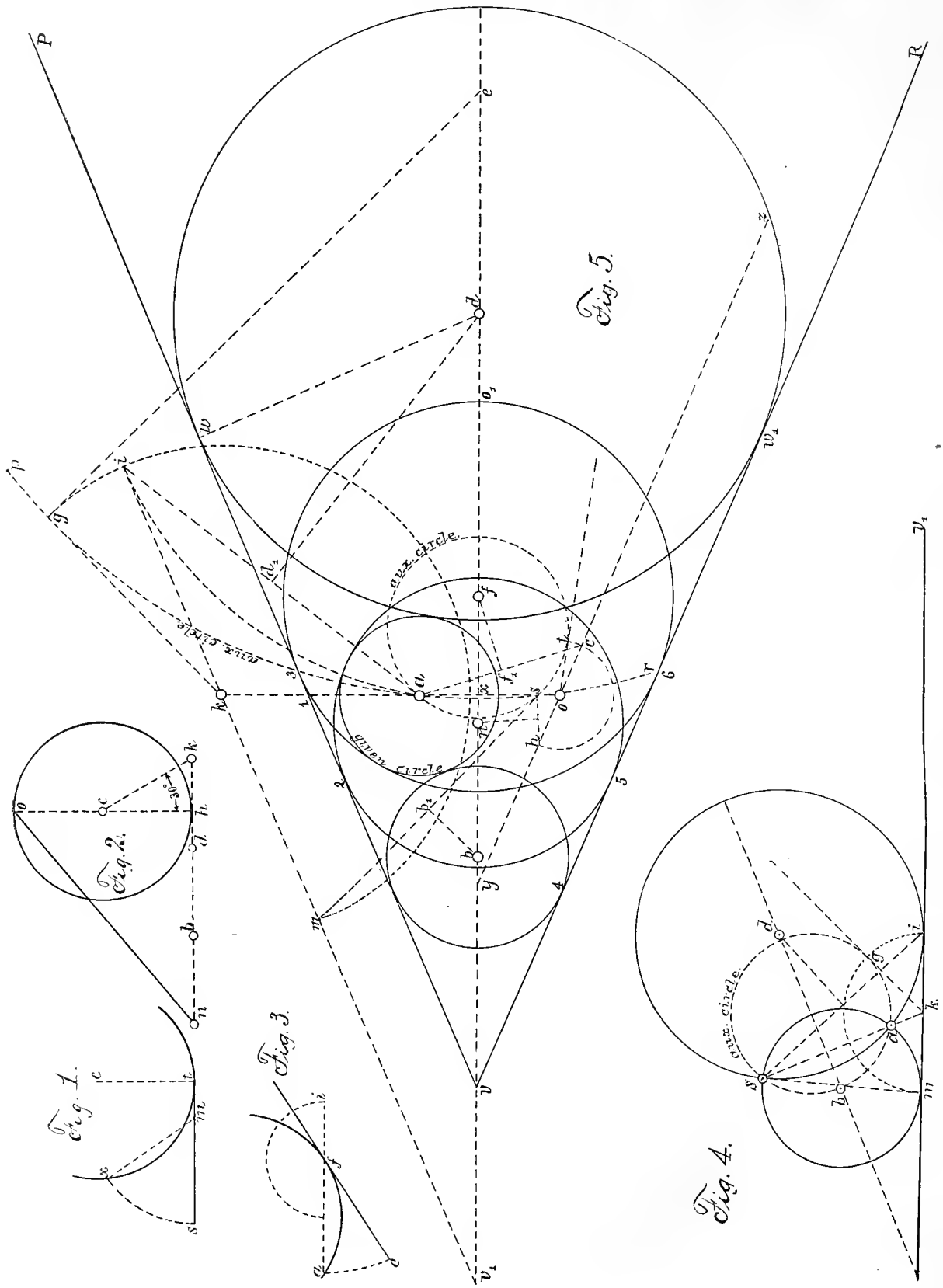


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
aux. circle
aux. circle
aux. circle
aux. circle
30°

cumference; care being taken to have the points of the dividers come exactly *on* the arc and also to avoid damaging the paper. If the seventh step goes past the starting point the dividers require closing; if it falls short, the original estimate was evidently too small. Obviously the change in setting the dividers ought in this case to be, as nearly as possible, *one-seventh* of the error; and after a few trials one should "come out even" on the last step.

102 *Prob. 17. To lay off on a given circle an arc of the same length as a given straight line.*[1] Let t (Plate I, Fig. 1) be one extremity of the desired arc; $t\,s$ the given straight line and tangent to the circle; $t\,m$ equal one-fourth of $t\,s$, and $s\,x$ drawn with centre m, radius $m\,s$. Then the length of the arc $t\,x$ is a close approximation to that of the line $t\,s$.

103. *Prob. 18. To lay off on a straight line the length of a given circular arc*[1] or, technically, to *rectify* the arc, let $a\,f$ (Plate I, Fig. 3) be the given arc; $a\,i$ the chord prolonged till $f\,i$ equals one-half the chord $a\,f$; and $a\,e$ an arc drawn with radius $a\,i$, centre i. Then $f\,e$ approximates closely to the length of the arc $a\,f$.

104. *Prob. 19. To obtain a straight line equal in length to a given semi-circle,*[2] draw a diameter $o\,h$ of the given semi-circle (Plate I, Fig. 2) and a radius inclined at an angle of 30° to the radius $e\,h$. Prolong the radius to meet the line $b\,h\,k$, drawn tangent to the circle at h. From k lay off the radius three times, reaching n. The line $n\,o$ equals the semi-circumference to four places of decimals.

105. *Prob. 20. To draw a circle tangent to two straight lines and a given circle.* (Four solutions.) This problem is given more on account of the valuable exercise it will prove to the student in absolute precision of construction than for its probable practical applications. Fig. 4 (Plate I) illustrates the geometrical principles involved and in it a circle is required to contain the points s and a and be tangent to the line $m\,r_1$. Draw first *any* circle containing s and a, as the one called "aux.

---

[1] These methods of approximation were devised by Prof. Rankine. They are sufficiently accurate for arcs not exceeding 60°. The error varies as the fourth power of the angle. The complete demonstration of Prob. 17 can be found in the Philosophical Magazine of October, 1867, and of Prob. 18 in the November issue of the same year.

[2] In his *Graphical Statics* Cremona states this to be the simplest method known for rectifying a semi-circumference. According to Böttcher it is due to a Polish Jesuit, Kochansky, and was published in the Acta Eruditorum Lipsiae, 1685.

Demonstration. Calling the radius *unity*, the diameter would have the numerical value 2.

$$\text{Then on} = \sqrt{\text{oh}^2 + \text{hn}^2} = \sqrt{\text{oh}^2 + (\text{kn} - \text{kh})^2} = \sqrt{4 + (3 - \tan 30°)^2} = 3.14153 +$$

The tangent of an angle (abbreviated to "tan.") is a trigonometric function whose numerical value can be obtained from a table. A draughtsman has such frequent occasion to use these functions that they are given here for reference, both as lines and as ratios.

Trigonometric Functions as Ratios.

θ = the given angle = $C\,A\,B$

h = hypothenuse of triangle $C\,A\,B$

$a = A\,B$ = side of triangle *adjacent* to vertex of θ

$o = B\,C$ = side of triangle *opposite* to θ

Then $\sin\,\theta = \frac{o}{h}$; $\cos\,\theta = \frac{a}{h}$;

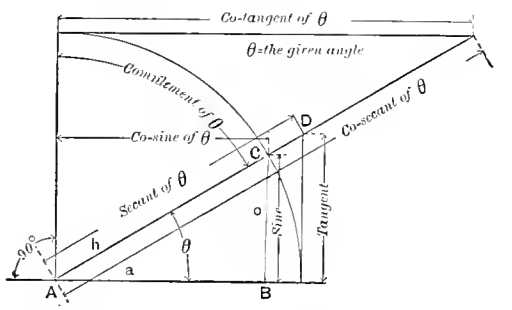
$\tan\,\theta = \frac{o}{a} = \frac{\sin\,\theta}{\cos\,\theta}$;

$\sec\,\theta = \frac{h}{a}$ = reciprocal of cosine.

$\operatorname{cosec}\,\theta = \frac{h}{o}$ " " sine

$\operatorname{cotan}\,\theta = \frac{a}{o} = \frac{\cos\,\theta}{\sin\,\theta}$ = reciprocal of tan θ.

Trigonometric Functions as Lines.



The prefix "co" suggests "complement," the *co*-sine of θ is the sine of the complement of θ, &c. As *lines*, the functions may be defined as follows:

The *sine* of an arc (e. g., that subtended by angle θ) is the perpendicular ($C\,B$) let fall from one extremity of the arc upon the diameter passing through the other extremity. If the radius A C, through one extremity of the arc, be prolonged to cut a line tangent at the other extremity, the intercepted portion of the tangent is called the *tangent* of the arc, and the distance, on such produced radius, from the centre of the circle to the tangent, is called the *secant of the arc.*

The co-sine, co-secant and co-tangent of the arc are respectively the sine, secant and tangent of the complement of the given arc.

circle." Join s to a and prolong to meet $m\,v_1$ at k. From k draw a tangent, $k\,g$, to the auxiliary circle. With radius $k\,g$ obtain m and i on the line $m\,v$. A circle through s, a and m or through s, a and i will fulfill the conditions. For $k\,g^2 = k\,s \times k\,a$, as $k\,g$ is a tangent and $k\,s$ a secant. But $k\,i = k\,g$ therefore $k\,i^2 = k\,s \times k\,a$ which would make $k\,i$ a tangent to a circle through s, a and i.

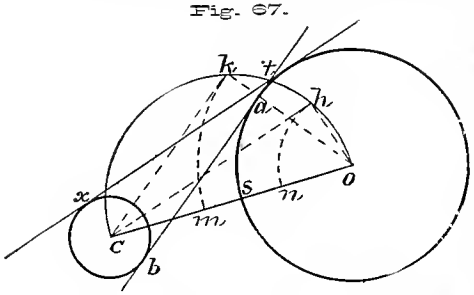
In Fig. 5 (Plate I) the construction is closely analogous to the above and the lettering identical for the first half of the work. The "given circle" is so called in the figure; the given lines are $P\,v$ and $R\,v$. Having drawn the bisector, $v\,e$, of the angle $P\,v\,R$, locate s as much below $v\,e$ as a (the centre of the given circle) is above it, the line $a\,s$ being perpendicular to $v\,e$. Draw $v_1\,m\,k\,i$ parallel to $v\,p$ and at a distance from it equal to the *radius* of the given circle. Then s, a, k and $m\,v_1$ of Fig. 5 are treated exactly as the analogous points of Fig. 4, and a circle obtained (centre d) containing a, s and i. The required circle will have the same centre d but radius $d\,w$, shorter than the first by the distance $w\,i$. Treat s, a, and m, (Fig. 5), similarly, getting the smallest of the four possible circles.

The remaining solutions are obtained by using the points a and s again but in connection with a line $y\,z$ parallel to $v\,R$ and *inside* the angle, again at a perpendicular distance from one of the given lines equal to the radius of the given circle.[1]

This problem makes a handsome plate if the *given* and *required* lines are drawn in *black;* the lines giving the first two solutions in *red;* and the remaining construction lines in *blue.*

106. *Prob. 21. To draw a tangent to two given circles* (a problem that may occur in connecting band-wheels by belts) join their centres, c and o (Fig. 67) and at s lay off $s\,m$ and $s\,n$ each equal to the radius of the smaller circle. Describe a semi-circle $o\,h\,k\,c$ on $o\,c$ as a diameter. Carry m and n to k and h, about o as a centre. Angles $c\,k\,o$ and $c\,h\,o$ are each 90°, being inscribed in a semi-circle; and $c\,k$ is parallel to $a\,b$, which last is one of the required tangents; while $c\,h$ is parallel to $t\,x$, a second tangent. Two more can be similarly found.

Fig. 67.



107. *Prob. 22. To unite two inclined straight lines by an arc tangent to both, radius given.* Prolong the given lines to meet at a (Fig. 68). With a as a centre and the given radius describe the arc $m\,n$. Parallels to the given lines and tangent to arc $m\,n$ meet at d, from which perpendiculars to the given lines give the points of tangency of the required arc which is then drawn with the given radius.

Fig. 68.

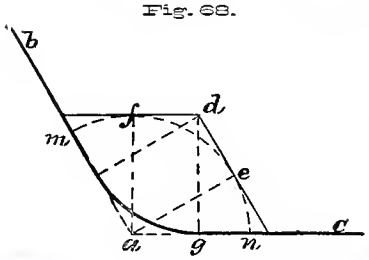
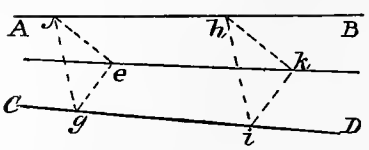


Fig. 69.



108. *Prob. 23. To draw through a given point a line which will—if produced—pass through the inaccessible*

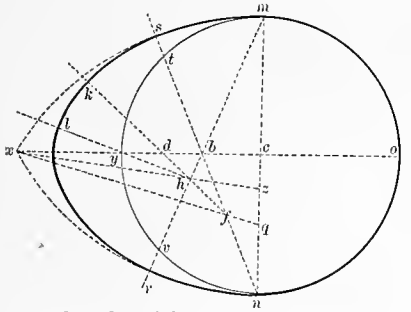
[1] This solution is taken from Benjamin Alvord's *Tangencies of Circles and of Spheres*, published by the *Smithsonian Institution*. That valuable pamphlet presents geometrical solutions of the ten problems of Apollonius on the tangencies of circles and also of the fifteen problems on the tangencies of spheres. These solutions are all based on the principle illustrated by Fig. 57, and applied in Prob. 20, that the tangent line or tangent curve is the limit of all secant lines or curves. The ten problems on the tangencies of circles, as proposed by the Greek geometer were the following:—

Prob. 1. To draw a circle through three points. One solution.
Prob. 2. Circle through two points and tangent to a given straight line. Two solutions.
Prob. 3. Circle through a given point and tangent to two straight lines. Two solutions.
Prob. 4. Circle through two points and tangent to a given circle. Two solutions.
Prob. 5. Circle through a given point, tangent to a given straight line and a given circle. Four solutions.
Prob. 6. Circle through a given point and tangent to two given circles. Four solutions.
Prob. 7. Circle tangent to three straight lines, two only of which may be parallel. Four solutions.
Prob. 8. Circle tangent to two straight lines and a given circle. Four solutions.
Prob. 9. Circle tangent to two given circles and a given straight line. Eight solutions.
Prob. 10. Circle tangent to three given circles. Eight solutions.

intersection of two lines. Join the given point e with *any* point f on $A\ B$ and also with some point g on $C\ D$. From any point h on $A\ B$ draw $h\ i$ parallel to $f\ g$, then $i\ k$ parallel to $g\ e$ and $h\ k$ parallel to $f\ e$. The line $k\ e$ will fulfill the conditions.

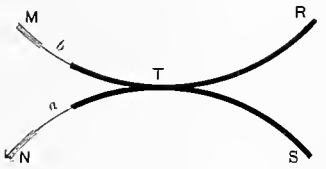
109. *Prob. 24. To draw an oval upon a given line.* Describe a circle on the given line, $m\ n$, (Fig 70) as a diameter. With m and n as centres describe arcs, $m\ x$, $n\ x$ radius $m\ n$. Draw $m\ v$ and $n\ t$ through v and t, the middle points of the quadrants $y\ m$, $y\ n$. Then $m\ s$ and $n\ r$ are the portions of $m\ x$ and $n\ x$ forming part of the oval. Bisect $n\ c$ at q and draw $q\ x$. Also bisect $c\ q$ at z and join the latter with x. Bisect $y\ b$ in d and draw $f\ d$ from f, the intersection of $n\ s$ and $q\ x$. Use f as a centre and $f\ s$ as radius for an arc $s\ k$ terminating on $f\ d$. The intersection, h, of $k\ f$ with $x\ z$ is then the next centre and $h\ k$ the radius of the arc $k\ l$ which terminates on $h\ y$ produced. The oval is then completed with y as a centre and radius $y\ l$. The lower portion is symmetrical with the upper and therefore similarly constructed.

Fig. 70.



110. Where exact tangency is the requirement novices occasionally endeavor to conceal a failure to secure the desired object by thickening the curve. Such a course usually defeats itself and makes more evident the error they thus hope to conceal. With such instruments of precision as the draughtsman employs there is but little, if any, excuse for overlapping or falling short.

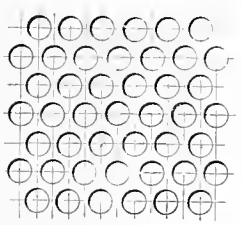
Fig 71.



A common error in drawing tangents where the lines are of appreciable thickness is to make the outsides of the lines touch; whereas they should have their thickness in common at the point of tangency, as at T (Fig. 70), where, evidently, the centre-lines a and b of the arcs would be *exactly* tangent, while the outer arc of M would come tangent to the inner arc of N.

111. When either a tube or a solid cylindrical piece is seen in the direction of its axis the outline is, evidently, simply a circle; and often the only way to determine which of the two the circle represented would be to notice which part of said end view was represented as casting a shadow. In Fig. 72, if the shaded arcs can cast shadows, the space *inside* the circles must be open, and the figure would represent a portion of the end view of a boiler with its tubular openings.

Fig. 72.



By exactly reversing the shading, the effect of which can be seen by turning the figure upside down, it is converted into a drawing of a number of solid, cylindrical pins projecting from a plate.

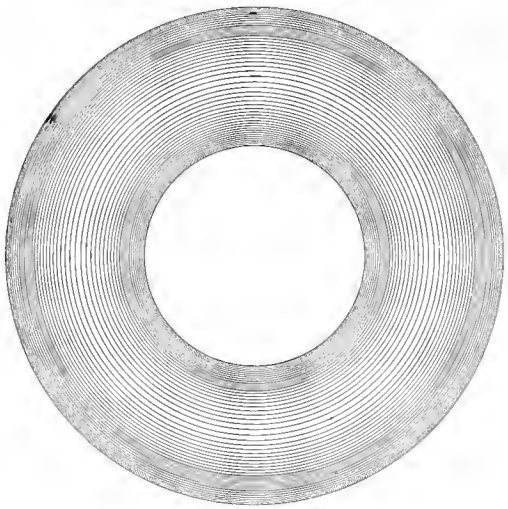
The tapering begins at the points where a diameter at 45° to the horizontal would cut the circumference.

To get a perfect taper on small circles use the bow-pen and after making one complete circle add the extra thickness by a second turn which is to begin with the pen-point *in the air*, the pen being brought down gradually upon the paper and then, while turning, raised from it again.

On medium and large circles the requisite taper can be obtained by a different process, viz., by using the same radius again but by taking a *second centre*, distant from the first by an amount equal to the proposed width of the broadest part of the shaded arc; the line through the two centres to be perpendicular to that diameter which passes through the extremities of the taper.

112. As an exercise in concentric circles Fig. 73 will prove a good test of skill. It is a fair representation of a gymnasium ring, the "annular torus" of mathematical works, and possessing

Fig. 73.



some remarkable properties, chief among which is the fact that it is the only surface of revolution known from which circles can be cut by three different systems of planes.*

Fig. 74.

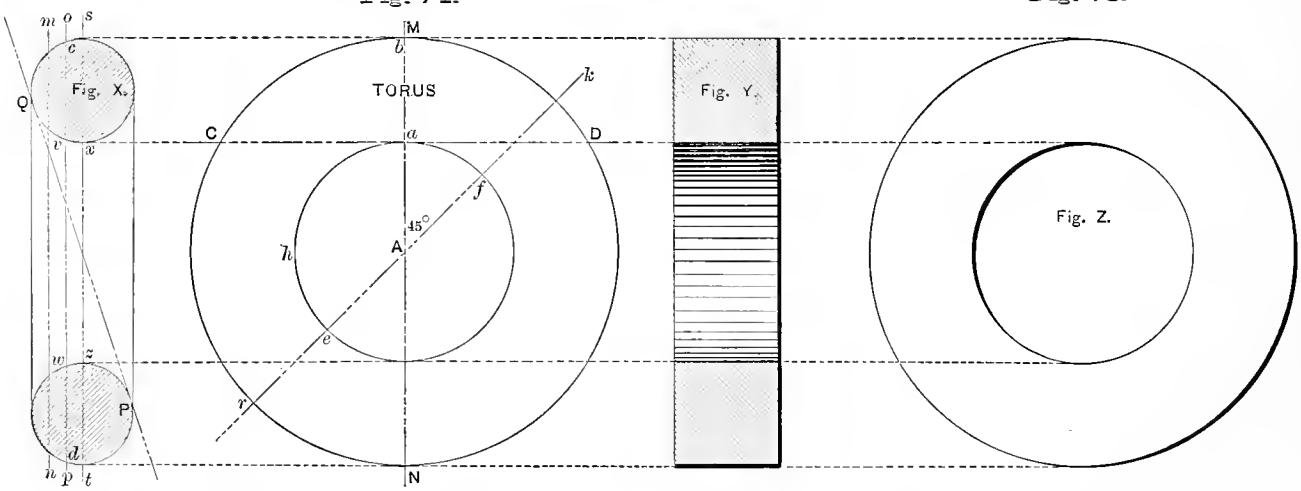


Fig. 75.

113. In Fig. 74 the same surface is shown, in the centre, in outline only. The axis of the surface would be a perpendicular to the paper at A. If $M\,N$ represents a plane perpendicular to the paper and containing the axis, then Fig. X will show the shape of the cut or *section*. As $M\,N$ was but one of the positions of a plane containing the axis and as the surface might be generated by rotating $M\,N$ with the circle $a\,b$ about the axis, it is evident that *one of the three systems of planes must contain the axis.*

When a surface can be generated by revolution about an axis one of its characteristics is that *any plane perpendicular to the axis will cut it in a circle.* The circles of Fig. 73 may then be, for the moment, considered as parallel cuts by a series of planes perpendicular to the axis, a few of which may be shown in $m\,n$, $o\,p$, &c. (Fig. X). Each of these planes cuts two circles from the surface; the plane $o\,p$, for example, giving circles of diameters $c\,d$ and $v\,w$ respectively.

* Olivier, *Memoires de Géométrie Descriptive.* Paris, 1851.

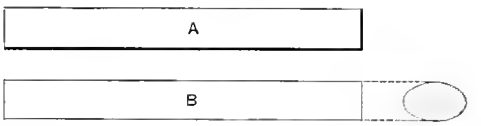
A plane perpendicular to the paper, on $P\,Q$, would be a *bi-tangent* plane, because tangent to the surface at two points, P and Q; and such plane would cut *two* over-lapping circles from the torus, each of them running partly on the inner and partly on the outer portion of the surface. For the proof that such sections are circles the student is probably not prepared at this point, but is referred to *Olivier's Seventh Memoir.*

114. Another interesting fact with regard to the torus is that a series of planes *parallel to, but not containing the axis,* cut it in a set of curves called the Cassian ovals, of which the Lemniscate of Art. 158 is a special case, and which would result from using a plane parallel to the axis and tangent to the surface at a point on the smallest circle, as at a, (Fig. 74.)*

115. Fig. Y is given to illustrate the fact that from mere untapered outlines, such as compose the central figure, we cannot determine the form of the object. By shading $e\,h\,f$ and $D\,N\,r$, the form shown in Fig. Y would be instantly recognized without the drawing of the latter. An angular object must therefore have shade lines, as also the *end view* of a round object; but a side view of a cylindrical piece must either have *uniform outlines* or be shaded with several lines.

Thus, in Fig. 76, *A* would represent an angular piece while *B* would indicate a circular cylinder; if *elliptical* its section would be drawn at one side as shown.

Fig. 76.



116. Before presenting the crucial test for the learner—the railroad rail—two additional practice exercises, mainly in ruling, are given in Figs. 77 and 78. The former shows that, like the parabola, the circle and hyperbola can be represented by their enveloping tangents. The upper and lower figures are merely two views of the surface called the *warped hyperboloid*, from the hyperbolas which constitute the curved outlines seen in the upper figure. The student can make this surface in a few moments by stringing threads through equidistant holes arranged in a circle on two circular discs, of the same or different sizes, but having the *same number of holes in each disc.* By attaching weights to the threads to keep them in tension at all times, and giving the upper disc a twist, the surface will change from cylindrical or conical to the hyperboloidal form shown.

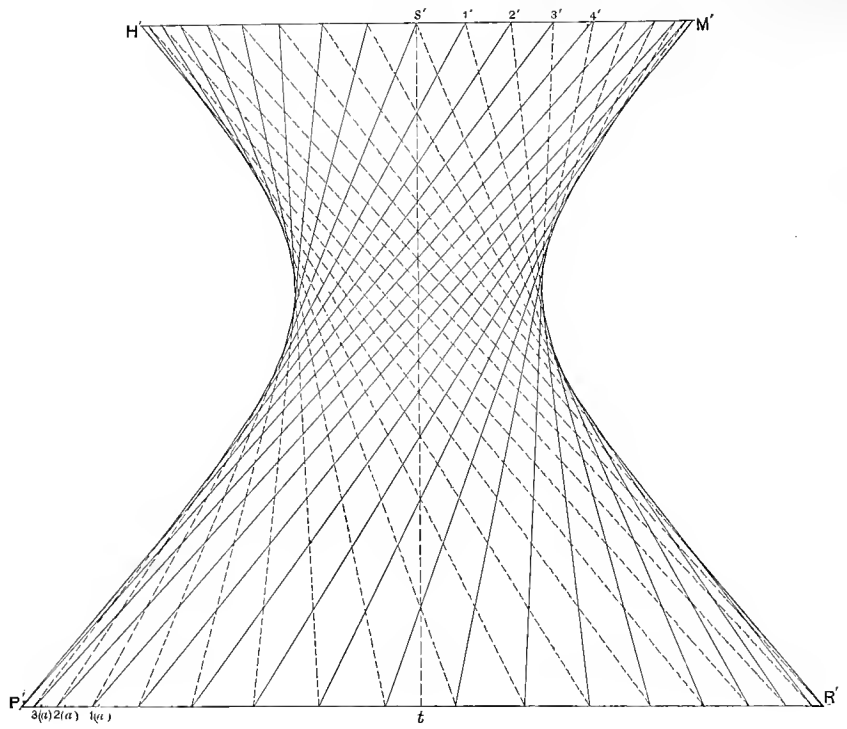
Gear wheels are occasionally constructed, having their teeth upon such a surface and in the direction of the lines or elements forming it; but the hyperboloid is of more interest mathematically than mechanically.

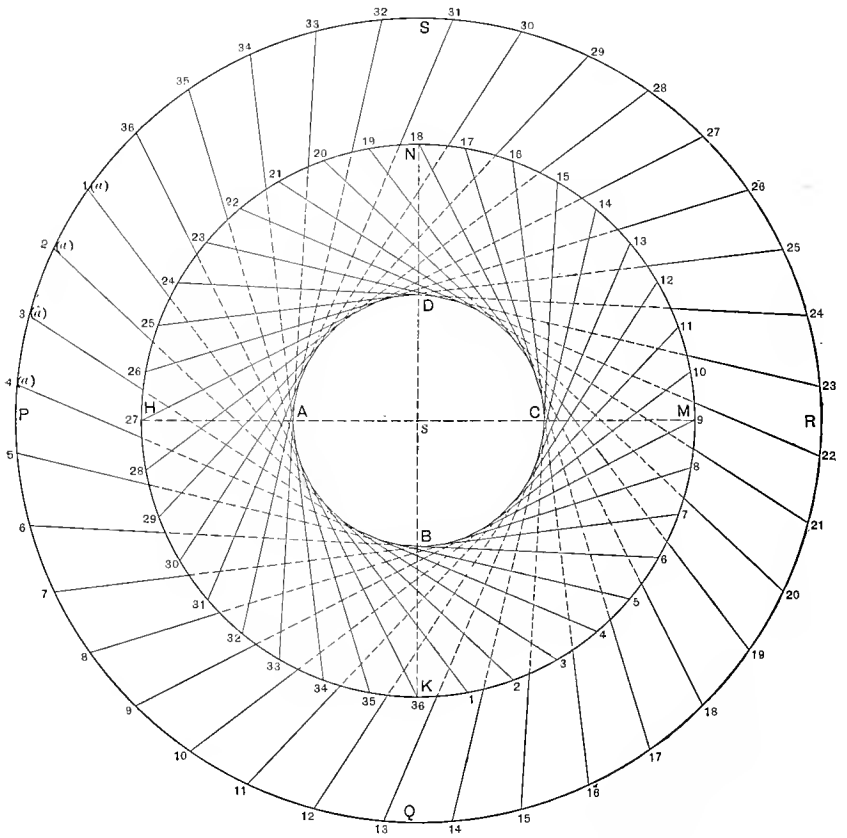
Begin the drawing by pencilling the three concentric circles of the lower figure. When inking, omit the smaller circle. Draw a series of tangents to the inner circle, each one beginning on the middle circle and terminating on the outer. Assume any vertical height, $t\,s'$, for the upper figure, and draw $H'\,M'$ and $P'\,R'$ as its upper and lower limits. $H'\,M'$ is the vertical projection, or elevation, of the circle $H\,K\,M\,N$, and *all* points on the latter as *1, 2, 3, 4,* are projected, by perpendiculars to $H'\,M'$, at $1', 2', 3', 4'$, etc. All points on the larger circle $P\,Q\,R$ are similarly projected to $P'\,R'$. The extremities of the same tangent are then joined in the upper view, as $1'$ with 1 (a).

Part of each line is dotted to represent its disappearing upon an invisible portion of the surface. The law of such change on the lower figure is evident from inspection; while on the eleva-

* These curves can also be obtained by assuming two foci, as if for an ellipse, but taking the *product* of the focal radii as a constant quantity, some perfect square. If $\rho\rho' = 36''$ then a point on the curve would be found at the intersection of arcs having the foci as centres, and for radii 2″ and 18″, or 4″ and 9″, etc. When the constant assumed is the square of half the distance between the foci, the Lemniscate results.

Fig. 77.



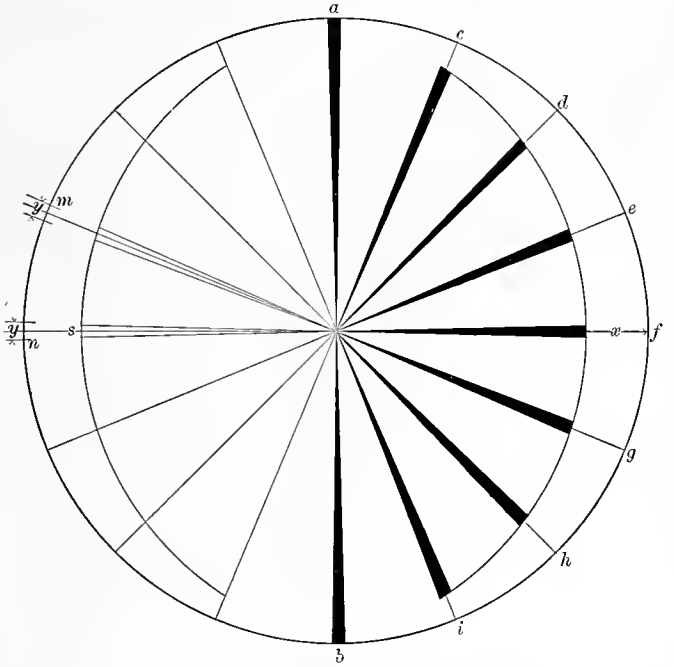


tion the point of division on each line is exactly above the point where the other view of the same line runs through *H M* in the lower figure.

117. To reproduce Fig. 78 draw first the circle *a f b n*, then two circular arcs which would contain *a* and *b* if extended, and whose greatest distance from the original circle is *x*, (arbitrary). Sixteen equidistant radii as at *a*, *c*, *d*, etc., are next in order, of which the rule and 45° triangle give those through *a*, *d*, *f* and *h*. At their extremities, as *m* and *n*, lay off the desired width, *y*, and draw toward the points thus determined lines radiating from the centre. Terminate these last upon the inner arcs. Ink by drawing *from* the centre, *not through* or *toward it*.

All construction lines should be erased before the tapering lines are filled in. The "filling in" may be done very rapidly by ruling the edges of the line in *fine* at first, then opening the pen slightly and beginning again where the opening between the lines is apparent and ruling from there, adding thickness to each edge on its inner side. It will then be but a moment's work to fill in, free-hand, with the Falcon pen or a fine-pointed sable-brush, between the now heavy edge-lines of the taper. To have the pen make a coarse line when starting from the centre would destroy the effect desired.

Fig. 78.



118. The draughtsman's ability can scarcely be put to a severer test on mere outline work than in the drawing of a railroad rail, so many are the changes of radii involved.

As previously stated, where tangencies to straight lines are required, the arcs are to be drawn first, then the tangents.

Figs. 79 and 80 are photo-engravings of rail sections, showing two kinds of "finish." Fig. 80 is a "working drawing" of a Pennsylvania Railroad rail, full-size. If finished with shade lines, as in Fig. 79, section-lined with Prussian blue, and the dimension lines drawn in carmine this makes one of the handsomest plates that can be undertaken.

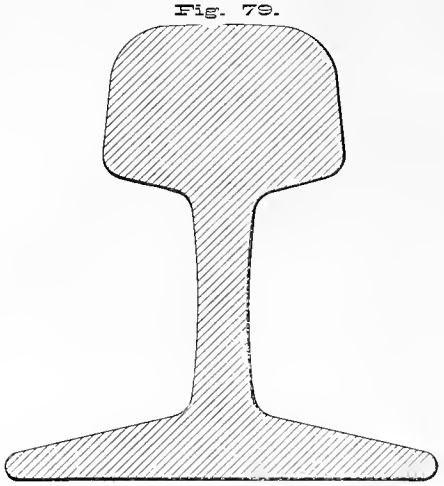
A still higher effect is shown in the wood-cut of the title-page, the rail being represented in oblique projection and shaded.

Begin Fig. 80 by drawing the vertical centre-line, it being an axis of symmetry. Upon it lay off 5″ for the total height, and locate two points between the top and base at distances from them of $1\frac{3}{4}$″ and $\frac{7}{8}$″ respectively; these to be the points of convergence of the lower lines of the head and sloping sides of the base. From these points draw lines, at first indefinite in length, and inclined 13° to the horizontal. The top of the head is an arc of 10″ radius, subtended by an angle of 9°. This changes into an arc of $\frac{7}{16}$″ radius on the upper corner, with its centre on the side of said 9° angle. The sides of the head are straight lines, drawn at 4° to the vertical and tangent to the corner arcs. The thin vertical portion of the rail is called the *web* and is $\frac{17}{32}$″ wide at its centre. The outlines of the web are arcs of 8″ radius, subtended by angles of 15°, centres on line marked "centre line of bolt holes."

The weight per yard of the rail shown is given as eighty-five pounds, from which we know the area of the cross-section to be eight and one-half square inches, since a bar of iron a yard long and one square inch in cross section weighs, approximately, ten pounds. (10.2 lbs., average.)

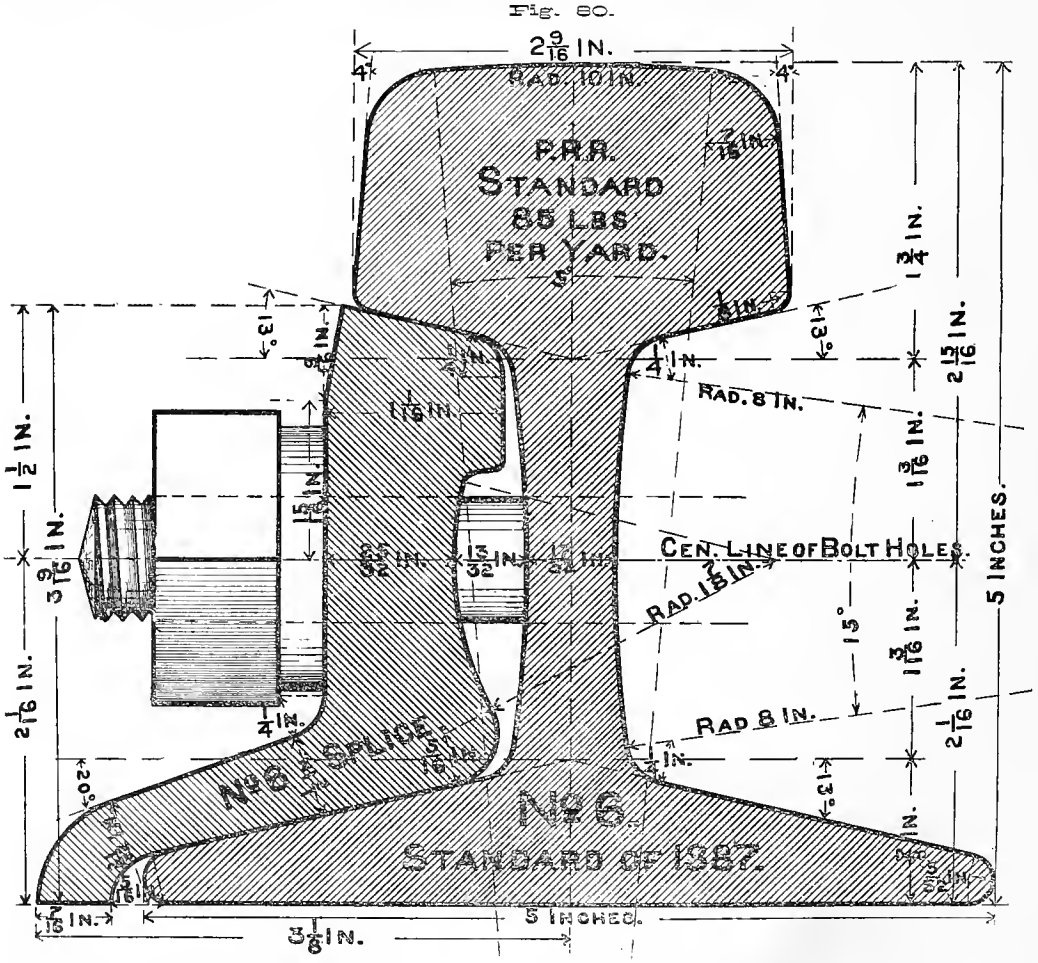
The proportions given are slightly different from those recommended in the report* of the committee appointed by the American Society of Civil Engineers to examine into the proper relations to

Fig. 79.



each other of the sections of railway wheels and rails. There was quite general agreement as to the following recommendations: a top radius of twelve inches; a quarter-inch corner radius; vertical sides

Fig. 80.



to the web; a lower-corner radius of one-sixteenth inch, and, lastly, a broad head relatively to the depth.

*Transactions A. S. C. E. January, 1891.

CHAPTER V.

THE HELIX.—CONIC SECTIONS.—HOMOLOGICAL PLANE CURVES AND SPACE-FIGURES.—LINK-MOTION CURVES.—CENTROIDS.—THE CYCLOID.—COMPANION TO THE CYCLOID.—THE CURTATE TROCHOID.—THE PROLATE TROCHOID.—HYPO-, EPI-, AND PERI-TROCHOIDS.—SPECIAL TROCHOIDS—ELLIPSE, STRAIGHT LINE, LIMAÇON, CARDIOID, TRISECTRIX, INVOLUTE, SPIRAL OF ARCHIMEDES.—PARALLEL CURVES.—CONCHOID.—QUADRATRIX.—CISSOID.—TRACTRIX.—WITCH OF AGNISI.—CARTESIAN OVALS.—CASSIAN OVALS.—CATENARY.—LOGARITHMIC SPIRAL.—HYPERBOLIC SPIRAL.—THE LITUUS.

119. There are many curves which the draughtsman has frequent occasion to make whose construction involves the use of the irregular curve. The more important of these are the Helix; Conic Sections—Ellipse, Parabola and Hyperbola; Link-motion curves or point-paths; Centroids; Trochoids; the Involute and the Spiral of Archimedes. Of less practical importance, though equally interesting geometrically, are the other curves mentioned in the heading.

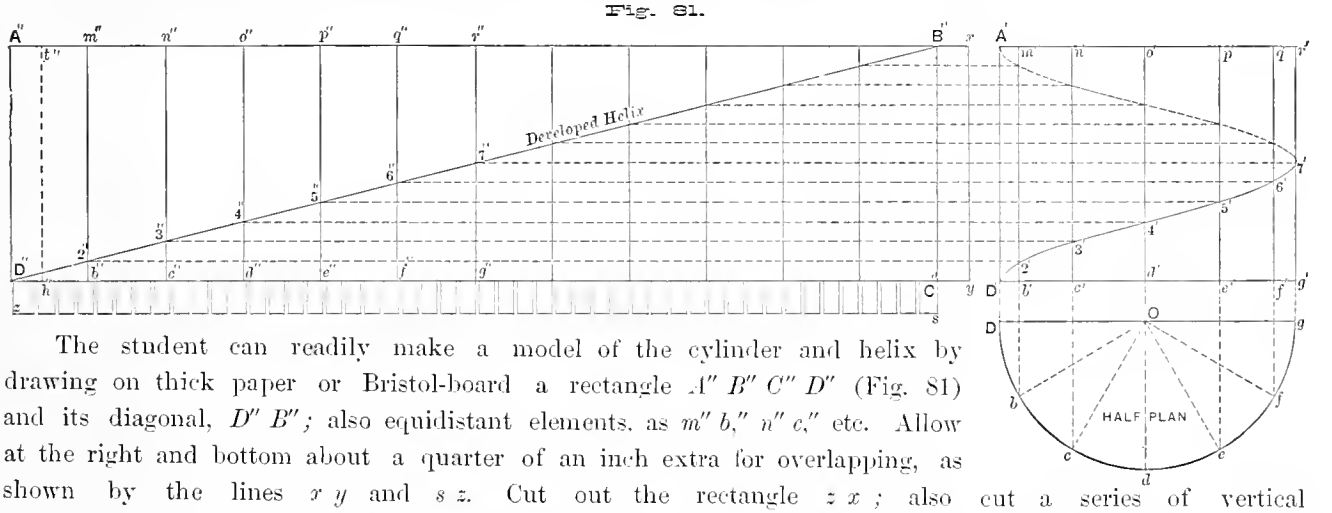
The student should become thoroughly acquainted with the more important geometrical properties of these curves, both to facilitate their construction under the varying conditions that may arise and also as a matter of education. Considerable space is therefore allotted to them here.

At this point Art. 58 should be reviewed, and in addition to its suggestions the student is further advised to work, at first, on as large a scale as possible, not undertaking small curves of sharp curvature until after acquiring some facility in the use of the curved ruler.

THE HELIX.

120. The ordinary helix is a curve which cuts all the elements of a cylinder at the same angle. Or we may define it as the curve which would be generated by a point having a uniform motion around a straight line combined with a uniform motion parallel to the line.

Fig. 81.



The student can readily make a model of the cylinder and helix by drawing on thick paper or Bristol-board a rectangle $A''\ B''\ C''\ D''$ (Fig. 81) and its diagonal, $D''\ B''$; also equidistant elements, as $m''\ b''$, $n''\ c''$, etc. Allow at the right and bottom about a quarter of an inch extra for overlapping, as shown by the lines $x\ y$ and $s\ z$. Cut out the rectangle $z\ x$; also cut a series of vertical slits between $D''\ C''$ and $z\ s$ and turn the divisions up at right angles to the paper; put mucilage between $B''\ C''$ and $x\ y$; then roll the paper up into cylindrical form, bringing $A''\ D''\ t''\ h''$ in front

of and upon the gummed portion, so that $A''\ D''$ will coincide with $B''\ C''$. The diagonal $D''\ B''$ will then be a helix on the outside of the cylinder, but half of which can be seen in front view, as $D'\ 7'$, (see right-hand figure); the other half, $7''\ A'$, being indicated as unseen.

To give the cylinder more permanent form it can be pasted to a cardboard base by mucilage on the under side of the divisions turned up along its lower edge.

The rectangle $A''\ B''\ C''\ D''$ is called the *development* of the cylinder; and any surface like a cylinder or cone, which can be rolled out on a plane surface and its equivalent area obtained by bringing consecutive elements into the same plane, is called a *developable surface.* The elements $m''\ b''$, $n''\ c''$, etc., of the development stand vertically at $b, c, d \ldots g$ of the half plan, and are seen in the elevation at $m'\ b'$, $n'\ c'$, $o'\ d'$, etc. The point $3'$, where any element, as c', cuts the helix, is evidently as high as $3''$ where the same point appears on the development. We may therefore get the curve $D'\ 7'\ A'$ by erecting verticals from $b, c, d, \ldots g$, to meet horizontals from the points where the diagonal $D''\ B''$ crosses those elements on the development.

The length $D''\ C''$ obviously equals $2\ \pi\ r$, in which $r = O\ D$.

The height $A'\ D'$, that the curve attains in winding once around the cylinder, is called the *pitch* of the helix.

The construction of the helix is involved in the designing of screws and screw-propellers, and in the building of winding stairs and skew-arches.

Mathematically the helix and its orthographic projection are well worth study, particularly the latter when the helix crosses the elements at 45°; it becoming then identical with Roberval's *curve of sines*, otherwise known as the *companion to the cycloid.*

For a helix on a cone refer to the article on the Spiral of Archimedes.

THE CONIC SECTIONS.

121. The ellipse, parabola and hyperbola are called *conic sections* or *conics* because they may be obtained by cutting a cone by a plane. We will, however, first obtain them by other methods.

According to the definition given by Boscovich, the ellipse, parabola and hyperbola are curves in which there is a *constant ratio* between the distances of points on the curve from a certain fixed point (the *focus*) and their distances from a fixed straight line (the *directrix*).

Referring to the parabola, Fig. 82, if S and B are points of the curve, F the focus and $X\ Y$ the directrix, then, if $S\ F{:}\ S\ T{::}\ B\ F{:}\ B\ X$, we conclude that B and S are points of a conic section.

122. The actual value of the ratio (or *eccentricity*) may be 1 or either greater or less than unity. When $S\ F$ equals $S\ T$ the ratio equals 1, and the relation is that of equality, or parity, which suggests the parabola.

123. If it is farther from a point of the conic to the focus than to the directrix the ratio is greater than 1 and the *hyperbola* is indicated.

124. The ellipse, of course, comes in for the third possibility as to ratio, viz., less than 1. Its construction by this principle is not shown in Fig. 82 but later, the method of generation here given illustrating the practical way in which, in landscape gardening, an elliptical plat would be laid out; it is therefore called the construction as the "gardener's ellipse."

Taking $A\ C$ and $D\ E$ as representing the extreme length and width, the points F and F_1 (*foci*) would be found by cutting $A\ C$ by an arc of radius equal to one-half $A\ C$, centre D. Pegs or pins at F and F_1, and a string, of length $A\ B$, with ends fastened at the foci, complete the preliminaries. The curve is then traced on the ground by sliding a pointed stake against the string, as at P, so that at all times the parts $F_1\ P$, $F\ P$, are kept straight.

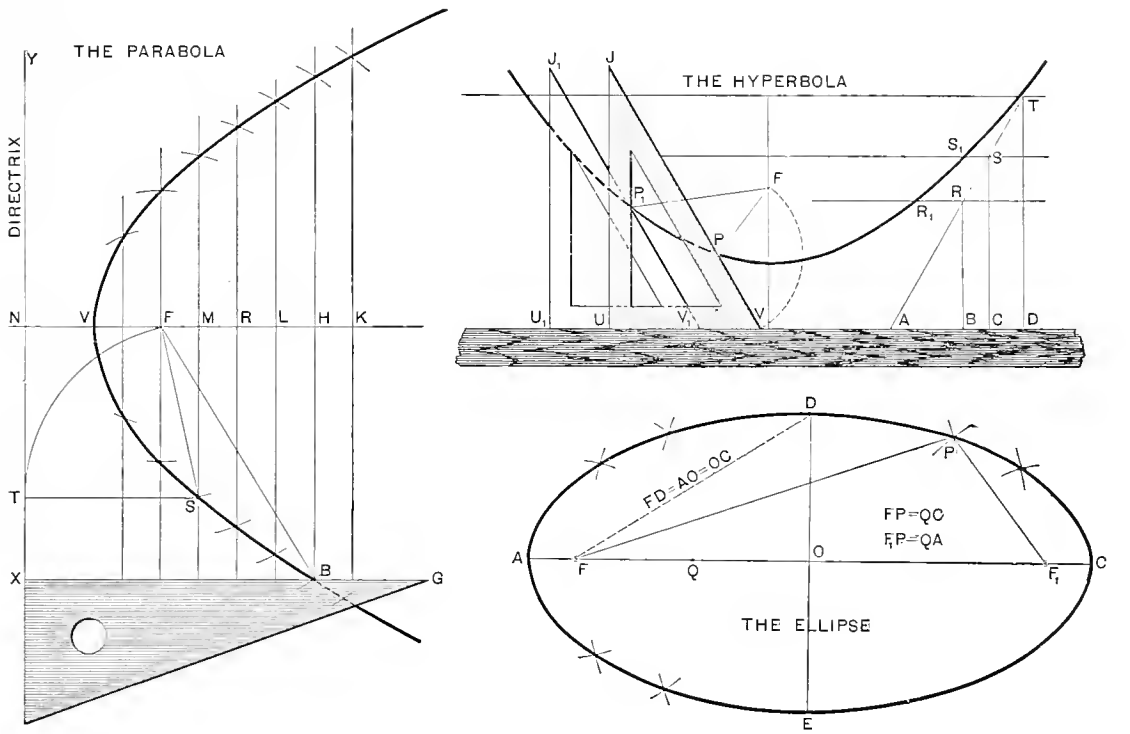
125. According to the foregoing construction the ellipse may be defined as a curve in which *the sum of the distances from any point of the curve to two fixed points is constant.* That *constant* is evidently the longer or *transverse* (*major*) axis, $A\ C$. The shorter or *conjugate* (or *minor*) axis, $D\ E$, is perpendicular to the other.

With the compasses we can determine P and other points of the ellipse by using F and F_1 as centres, and for radii any two segments of $A\ C$. Q, for example, gives $A\ Q$ and $C\ Q$ as segments. Then arcs from F and F_1, with radius equal to $Q\ C$, would intersect arcs from the same centres, radius $Q\ A$, in four points of the ellipse, one of which is P.

126. By the Boscovich definition, we are enabled to construct the parabola and hyperbola also by continuous motion along a string.

For the parabola place a triangle as in Fig. 82, with its altitude $G\ X$ toward the focus. If a string *of length* $G\ X$ be fastened at G, stretched tight from G to *any* point B, by putting a pencil at B, then the remainder $B\ X$ swung around and the end fastened at F, it is then, evidently, as far from B to F as it is from B to the directrix; and that relation will remain constant as the triangle is slid along the directrix, if the pencil point remains against the edge of the triangle so that the portion of the string from G to the pencil is kept straight.

Fig. 82.



127. For the hyperbola, (Fig. 82), the construction is identical with the preceding, except that the string fastened at J runs down the *hypothenuse* and equals it in length.

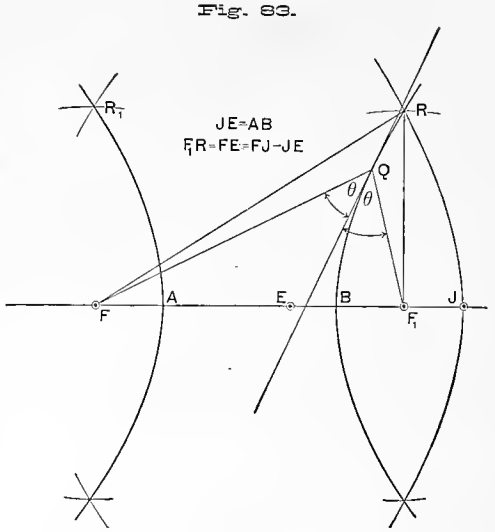
128. Referring back to Fig. 35, it will be noticed that the focus and directrix of the parabola are there omitted; but the former would be the point of intersection of a perpendicular from A upon the line joining C with E. A line through A, parallel to $C\ E$, would be the directrix.

129. Like the ellipse, the hyperbola can be constructed by using two foci, but whereas in the ellipse (Fig. 82) it was the *sum* of two focal radii that was constant, i. e., $F\ P + F_1\ P = F\ D +$

$F_1 D = A C$ (the transverse axis), it is the *difference* of the radii that is constant for the hyperbola.

In Fig. 83, if $A B$ is to be the transverse axis of the two arcs, or "branches," which make the complete hyperbola, then using ρ and ρ' to represent *any* two focal radii, as $F Q$ and $F_1 Q$, or $F R$ and $F_1 R$, we will have $\rho - \rho' = A B$ (the constant quantity).

Fig. 83.



To get a point of the curve in accordance with this principle we may lay off from either focus, as F, any distance greater than $F B$, as $F J$, and with it as a radius, and F as a centre, describe the indefinite arc $J R$. Subtracting the constant, $A B$, from $F J$, by making $J E = A B$, we use the remainder, $F E$, as a radius, and F_1 as a centre to cut the first arc at R. The same radii will evidently determine three other points fulfilling the conditions.

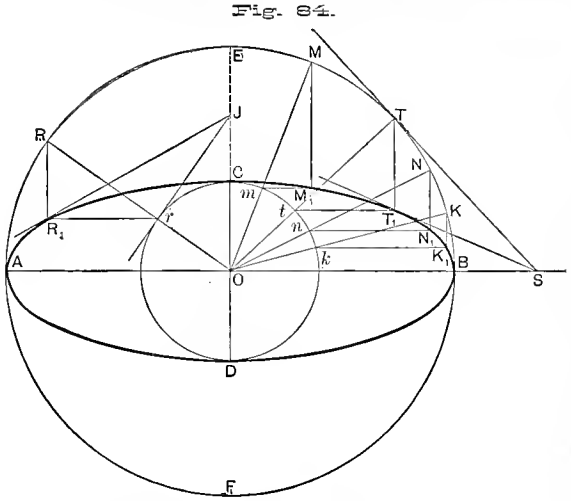
130. The tangent to the hyperbola at any point, as Q, bisects the angle $F Q F_1$, between the focal radii.

In the ellipse, (Fig. 82), it is the *external* angle between the radii that is bisected by the tangent.

In the parabola, (Fig. 82), the same principle applies, but as one focus is supposed to be *at infinity*, the focal radius, $B G$, toward the latter, from any point, as B, would be parallel to the axis. The tangent at B would therefore bisect the angle $F B X$.

131. *The ellipse as a circle viewed obliquely.* If $A R M B F$ (Fig. 84) were a circular disc and we were to rotate it on the diameter $A B$, it would become narrower in the direction $F E$ until, if sufficiently turned, only an edge view of the disc would be obtained. The axis of rotation, $A B$, would, however, still appear of its original length. In the rotation supposed, all points not on the axis would describe circles about it with their planes perpendicular to it. M, for example, would move in the arc of a circle part of which is shown in $M M_1$, which is straight, as the plane of the arc is seen "edge-wise." If instead of a circular disc we were to take one of *elliptical* form, as $A C B D$, and turn it upon its shorter axis $C D$, it is obvious that B would apparently approach O on one side while A advanced on the other; and when B was directly in front of, and projected at k, we would have the ellipse projected in the small circle $r\ t\ k$. Having given then the two axes of a desired ellipse, as $A B$ and $C D$, use them as diameters of concentric circles and from their common centre, O, draw random radii, as $O M$, $O T$, $O K$. Where any radius, $O M$, meets the outer circle, drop a perpendicular $M M_1$ to the longer axis, to meet a line $m M_1$, parallel to the same longer axis and passing through m, where $O M$ cuts the smaller circle.

Fig. 84.

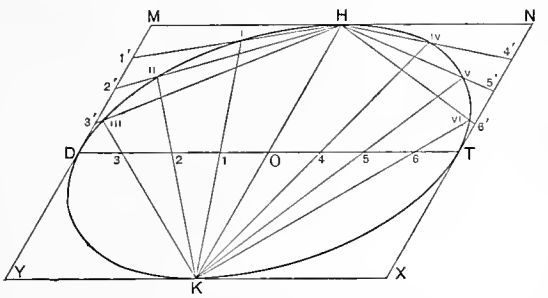


132. If $T S$ is a tangent at T to the large circle, then when T appears at T_1 we shall have $T_1 S$ as a tangent to the ellipse at the point derived from T; S having remained constant, being on the axis of rotation.

Similarly, if a tangent at R_1 were wanted we may first find r, corresponding to R_1; draw the tangent $r\,J$ to the small circle; then join J, (constant point), with R_1.

133. Occasionally we have given the length and inclination of a pair of diameters of the ellipse, making oblique angles with each other. Such diameters are called *conjugate* and the curve may be constructed upon them thus: Draw the axes $T\,D$ and $H\,K$ at the assigned angle $D\,O\,H$; construct the parallelogram $M\,N\,X\,Y$; divide $D\,M$ and $D\,O$ into the same number of equal parts; then from K draw lines through the points of division on $D\,O$, to meet similar lines through H and the divisions on $D\,M$. The intersection of like numbered lines will give points of the ellipse.

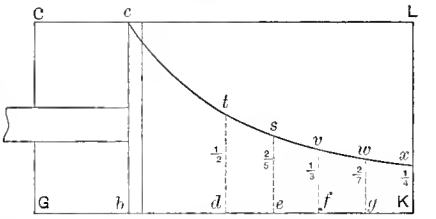
Fig. 85.



134. It is the law of expansion of a perfect gas that the volume is inversely as the pressure. That is, if the volume be doubled the pressure drops one-half; if trebled the pressure becomes one-third, etc. Steam not being a perfect gas departs somewhat from the above law, but the curve indicating the fall in pressure due to its expansion is compared with that for a perfect gas.

To construct the curve for the latter let us suppose $C\,L\,K\,G$ (Fig. 86) to be a cylinder with a volume of gas $C\,G\,b\,c$ behind the piston. Let $c\,b$ indicate the pressure before expansion begins. If the piston be forced ahead by the expanding gas until the volume is doubled, the pressure will drop, by Boyle's law, to one-half and will be indicated by $t\,d$. For three volumes the pressure becomes $v\,f$, etc. The curve $c\,s\,x$ is an hyperbola, the special case called *equilateral*.

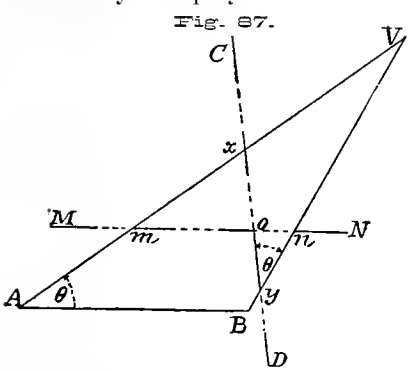
Fig. 86.



Suppose the cylinder were infinite in length. Since we cannot conceive a volume so great that it could not be doubled, or a pressure so small that it could not be halved, it is evident that theoretically the curve $c\,x$ and the line $G\,K$ will forever approach each other yet never meet; that is, they will be *tangent at infinity*. In such a case the straight line is called an *asymptote* to the curve.

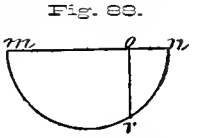
135. Although the *right* cone (i. e., one having its axis perpendicular to the plane of its base) is usually employed in obtaining the ellipse, hyperbola and parabola, yet the same kind of sections can be cut from an oblique or *scalene* cone of circular base, as $V\,A\,B$, Fig. 87. Two sets of *circular* sections can also be cut from such a cone, one set, obviously, by planes parallel to the base, the other by planes like $C\,D$, making the same angle with the lowest element, $V\,B$, that the highest element, $V\,A$, makes with the base. The latter sections are called *sub-contrary*.

Fig. 87.



To prove that the section $x\,y$ is a circle we note that both it and the section $m\,n$—the latter known to be a circle because parallel to the base—intersect in a line perpendicular to the paper at o. This line pierces the front surface of the cone at a point we may call r. It would be seen as the ordinate $o\,r$ (Fig. 88), were the front half of circle $m\,n$ rotated until parallel to the paper. Then $o\,r^2 = o\,m \times o\,n$. But from the similar triangles in Fig. 87 we have $o\,m : o\,y :: o\,x : o\,n$ whence $o\,y \times o\,x = o\,m \times o\,n = o\,r^2$, thus showing that the section $x\,y$ is circular.

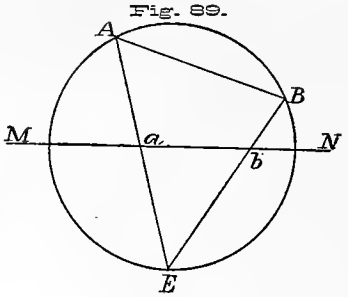
Fig. 88.



Were the vertex of a scalene cone removed *to infinity* the cone would become an oblique cylinder with circular base; but the latter would possess the property just established for the former.

136. The most interesting practical application of the sub-contrary section is in Stereographic Projection, one of the methods of representing the earth's surface on a map. The especial convenience of this projection is due to the fact that in it every circle is projected as a circle. This results from the relative position of the eye (or centre of projection) and the plane of projection; the latter is that of some great circle of the earth and the eye is located at the pole of such circle.

Fig. 89.



137. In Fig. 89 let the circle $A\ B\ E$ represent the equator; $M\ N$ the plane of a meridian, also taken as the plane of projection; $A\ B$

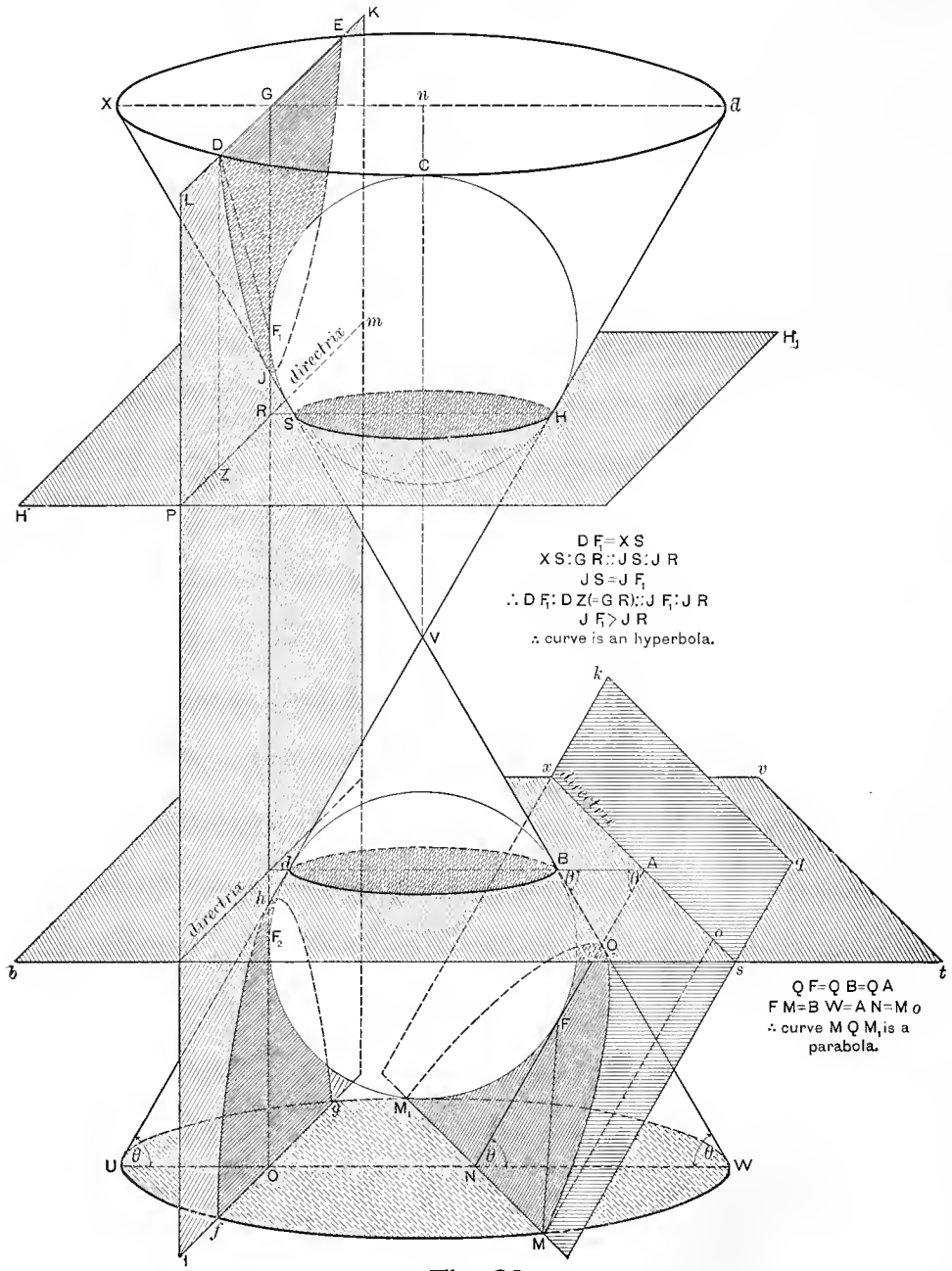


Fig. 90.

any circle of the sphere; E the position of the eye: then $a\ b$, the projection of $A\ B$ on plane $M\ N$,

(see Art. 4), is a circle, being a sub-contrary section of the visual cone $E.A\,B$, as the student can easily prove.

138. We now take up the conic sections as derived from a right cone.

A complete cone (Fig. 90) lies as much above as below the vertex. To use the term adopted from the French, it has two *nappes*.

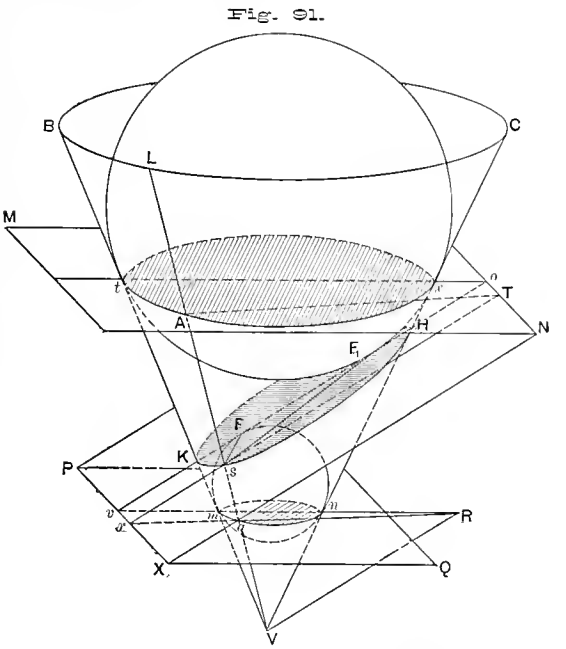
Aside from the extreme cases of *perpendicularity to* or *containing* the axis *the inclination of a plane cutting the cone may be*

(a) *Equal to* that of the elements,* therefore parallel to *one* element, giving the *parabola*, as $M\,Q\,M_1$ (Fig. 90); the plane $k\,q\,M$ being parallel to the element $V\,U$ and therefore making with the base the same angle, θ, as the latter.

(b) *Greater than* that of the elements, causing the plane to cut both nappes and giving a two-branch curve, the *hyperbola*, as $D\,J\,E$ and $f\,h\,g$ (Fig. 90).

(c) *Less than* that of the elements, the plane therefore cutting all the elements on one side of the vertex, giving a closed curve, the ellipse; as $K\,s\,H$, Fig. 91.

Fig. 91.



139. Figures 90 and 91, with No. 4 of Plate 2 are not only stimulating examples for the draughtsman but they illustrate probably the most interesting fact met with in the geometrical treatment of conic sections, viz., that the spheres which are tangent simultaneously to the cone and the cutting planes, touch the latter in the foci of the conics; while in each case the directrix of the curve is the line of intersection of the cutting plane and the plane of the circle of tangency of cone and sphere.

To establish this we need only employ the well known principles that (a) all tangents from a point to a sphere are equal in length, and (b) all tangents are equal that are drawn to a sphere from points equidistant from its centre. In both figures all points of the cone's bases are evidently equidistant from the centres of the tangent spheres.

140. On the upper nappe, (Fig. 90), let $S\,H$ be the circle of contact of a sphere which is tangent at F_1 to the cutting plane $P\,L\,K$. The plane $P\,H_1$, of the circle cuts the plane of section in $P\,m$. If D is *any* point of the curve $D\,J\,E$, J another point, and we can prove the ratio constant (and greater than unity) between the distances of D and J from F_1 and their distances to $P\,m$, then the curve $D\,J\,E$ must be an *hyperbola*, by the Boscovich definition; F_1 must be the focus and $P\,m$ the directrix.

$D\,F_1$ is a tangent whose real length is seen at $X\,S$. $J\,F_1$ and $J\,S$ are equal, being tangents to the sphere from the same point. We have then the proportion $X\,S : G\,R :: J\,S : J\,R$ or $D\,F_1 : D\,Z :: J\,F_1 : J\,R$. Since $J\,S$ and its equal $J\,F_1$ are greater than $J\,R$, and the ratio $J\,F_1$ to $J\,R$ is constant, the proposition is established.

141. For the parabola on the lower nappe, since the plane $M\,e\,k$ is inclined at the angle θ made by the elements, we have $Q\,A = Q\,B$ (opposite equal angles) and $Q\,B$ equal $Q\,F$ (equal tangents). $M\,F = B\,W = M\,o$, therefore $M\,F : M\,o :: Q\,F : Q\,o$, and it is as far from M to the focus F as to the directrix $s\,x$, fulfilling the condition essential for the parabola.

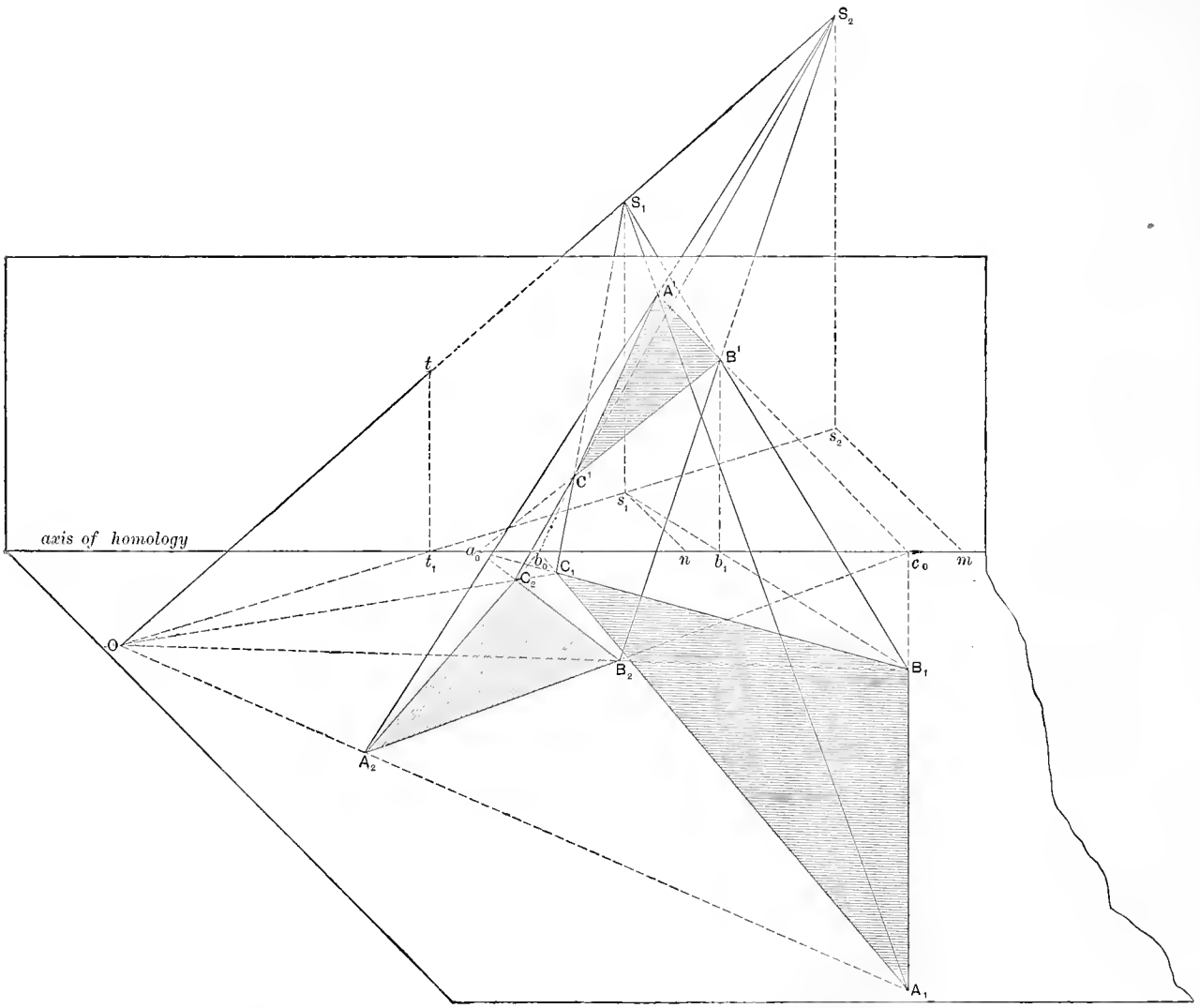
*See Remark, Art. 4.

142. For the ellipse $K\,s\,H$, (Fig. 91), we have $P\,N$ and $N\,T$ as the lines to be proven directrices, and F and F_1 the points of tangency of two spheres. Let s be any point of the curve under consideration and $V\,L$ the element containing s. This element cuts the contact circles of the spheres in a and A. A plane through the cone's axis and parallel to the paper would contain $o\,t$, $o\,v$ and $v\,n$. Prolong $v\,n$ to meet a line $V\,R$ that is parallel to $K\,H$. Join R with a, producing it to meet $P\,N$ at r. In the triangles $a\,s\,r$ and $a\,V\,R$ we have $s\,a : s\,r :: V\,a : V\,R$. But $s\,a = s\,F$ (equal tangents) and similarly $V\,a = V\,n$; therefore $s\,F : s\,r :: V\,n : V\,R$, which ratio is less than unity; therefore a is a point on an ellipse.*

The plane of the intersecting lines $V\,a$ and $R\,r$ cuts the plane $M\,N$ in $A\,T$ which is therefore parallel to $a\,r$; therefore $s\,A : s\,T :: V\,n : V\,R$. But $s\,A = s\,F_1$ therefore $s\,F_1 : s\,T :: V\,n : V\,R$, the same ratio as before.

143. If the plane of section $P\,N$ were to approach parallelism to $V\,C$ the point R would advance toward n, and when $V\,R$ became $V\,n$ the plane would have reached the position to give the parabola.

Fig. 92.



* Schlömilch, *Geometrie des Maasses*, 1874.

144. The proof that $K\,s\,H$ (Fig. 91) is an ellipse when the curve is referred to *two foci* is as follows: $K\,F = K\,m$; $K\,F_1 = K\,t$; therefore $K\,F + K\,F_1 = K\,m + K\,t = t\,m = x\,n = 2\,K\,F + F\,F_1 = 2\,H\,F + F\,F_1$; i. e., $K\,F = H\,F_1$.

Since $s\,F = s\,a$ and $s\,F_1 = s\,A$ we have $s\,F + s\,F_1 = s\,a + s\,A = A\,a = t\,m = x\,n = H\,K$. The sum of the distances from any point s to the two fixed points F and F_1 is therefore constant and equal to the longer axis $H\,K$.

HOMOLOGOUS PLANE AND SPACE FIGURES.

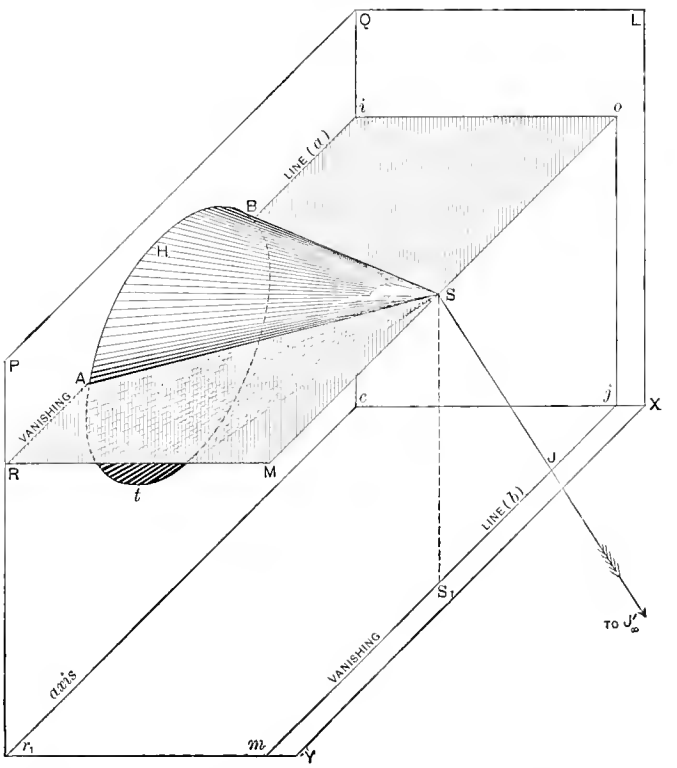
145. Before leaving the conic sections their construction will be given by the methods of Projective Geometry. (See Art. 9.)

In Fig. 92, if S_1 is a centre of projection, then, by Art. 4, the figure $A^1\,B^1\,C^1$ is the *central projection* of $A_1\,B_1\,C_1$. The points A^1 and A_1 are *corresponding points*. Similarly B^1 and B_1, C^1 and C_1.

For S_2 as the centre of projection the figure $A_2\,B_2\,C_2$ corresponds to $A^1\,B^1\,C^1$.

If we join S_2 with S_1 and prolong to O, to meet the plane of the figures $A_1\,B_1\,C_1$ and $A_2\,B_2\,C_2$, we find that the point O sustains the same relation to these two figures that S_1 and S_2 do to the figures just taken in connection with them. Using the technical term *trace* for the intersection of a line with a plane or of one plane with another, we see that $A^1\,c_0$ is the trace, on the vertical plane, of any plane containing $A^1\,B^1$. This plane cuts the "axis of homology," $t_1\,m$, in c_0. As $A_1\,B_1$ lies in the plane of S_1 and $A^1\,B^1$, and in the horizontal plane as well, it can only meet the vertical plane in c_0, the point of intersection of all these planes. Similarly we find that $A^1\,C^1$ and $A_1\,C_1$, if prolonged, meet the axis at the same point; correspondingly $B^1\,C^1$ and $B_1\,C_1$ meet at a_0. But $A^1\,B^1$ and $A_2\,B_2$, being corresponding lines, lie in the plane with S_2, though belonging to figures in two other planes; they must, therefore, meet also at the same point, c_0; and similarly for the other lines in the figures used with S_2.

Fig. 92.



Finally, the line $A_1\,A_2$ being the horizontal trace of the plane determined by the lines joining A^1 with S_1 and S_2 must contain the horizontal trace, O, of the line joining S_2 with S_1. But this puts A_2 and A_1 into the same relation with O that A_2 and A^1 sustain to S_2; or that of A^1 and A_1 to S_1. Hence we rightly conclude that *on one plane* we can take a point O as a centre of projection, and a figure $A_2\,B_2\,C_2$, and from them derive a second figure, $A_1\,B_1\,C_1$, which *corresponds* to the assumed figure in the same way as if they lay in different planes. Figures so related in one plane are called *homological figures* and the centre, O, a *centre of homology*.

146. Had $A_1\,B_1\,C_1$ been a *circle*, and all its points joined with S_1, then from what has preceded we know that $A^1\,B^1\,C^1$ would have been an ellipse; as also would have been the case were $A_2\,B_2\,C_2$ a circle used in connection with S_2. But our conclusions should enable us to substitute a circle for $A_2\,B_2\,C_2$ and using

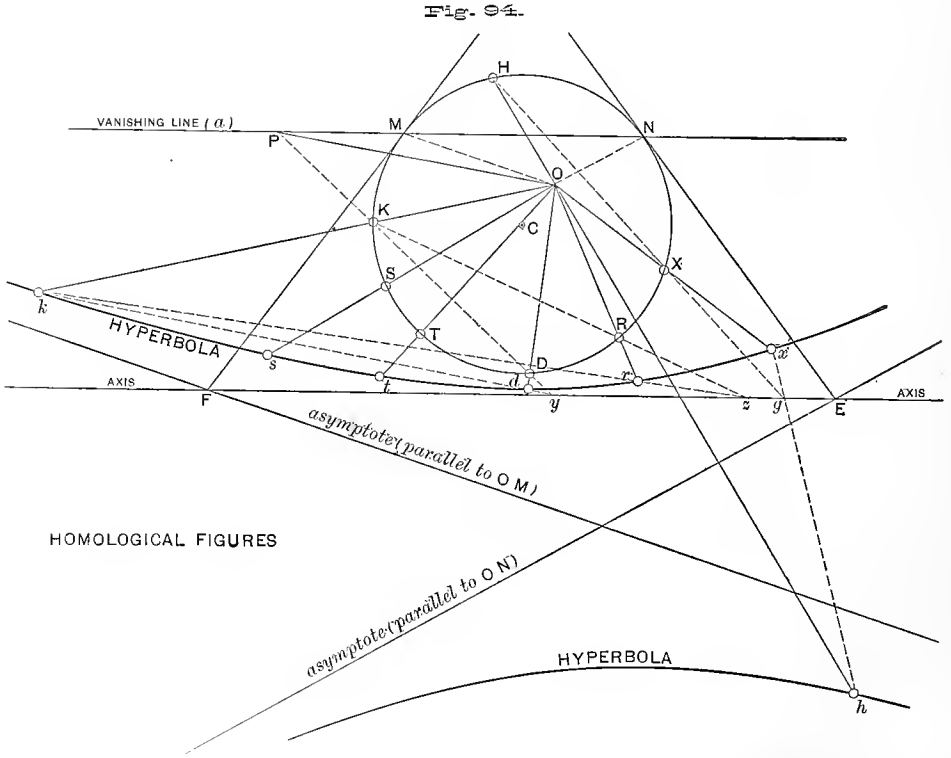
O in the same plane with it get an ellipse in place of the triangle $A_1 B_1 C_1$. Before illustrating this it is necessary to show the relation of the axis to the other elements of the problem and to supply a test as to the nature of the conic.

147. First as to the *axis*, and employing again for a time a space figure (Fig. 93), it is evident that raising or lowering the horizontal plane $c\,X\,Y$ parallel to itself, and with it, necessarily, the axis, would not alter the *kind* of curve that it would cut from the cone $S.\,H\,A\,B$, were the elements of the latter prolonged. But raising or lowering the *centre*, S, would decidedly affect the curve. Where it is, there are two elements of the cone, $S\,A$ and $S\,B$, which would never meet the plane $c\,X\,Y$. The shaded plane containing those elements meets the vertical plane in "vanishing line (a)," parallel to the axis. This contains the projections, A and B, of the points at infinity where the lower plane may be considered as cutting the elements $S\,A$ and $S\,B$. Were S and the shaded plane raised to the level of H, so that "vanishing line (a)" should become tangent to the base, there would be one element, $S\,H$, of the cone, parallel to the lower plane, and the section of the cone by the latter would be the parabola; while the figure as it stands would indicate the hyperbola. The former would have but one point at infinity; the latter, two.

148. Raise the centre S so that the vanishing line does not cut the base and, evidently, no line from S to the base would be parallel to the lower plane; but the latter would cut all the elements on one side of the vertex, giving the ellipse.

149. Bearing in mind that the projection of the circle $A\,H\,B\,t$ is on the lower plane produced, if we wish to bring both these figures and the centre S into one plane, without destroying the relation between them, we may imagine the end plane $Q\,L\,X$ removed and rotation of the remaining system

Fig. 94.



occurring about $c\,r_1$ in a manner exactly similar to that which would occur were $i\,o\,j\,c$ a system of four pivoted links and the point o pressed toward c. The motion of S would be parallel and equal

to that of o, and, like the latter, would evidently maintain its distance from the vanishing line and describe a circular arc about it. The vanishing line would remain parallel to the axis.

150. From the foregoing we see that to obtain the hyperbola by projection of a circle from a point in the plane of the latter we would require simply a secant vanishing line, $M\,N$ (Fig. 94), and an axis of homology parallel to it. Take any point P on the vanishing line and join it with any point K of the circle. $P\,K$ meets the axis at y; therefore the line $k\,y$ that *corresponds* to $P\,K$ must also meet the axis at y. $O\,P$ is analogous to $S\,A$ of Fig. 93 in that it meets its corresponding line at infinity, i. e., is parallel to it. Therefore $y\,k$, parallel to $O\,P$ meets the ray $O\,K$ at k, corresponding to K. Then K joined with any other point R gives $K\,z$. Join z with the point k just obtained and prolong $O\,R$ to intersect them, obtaining r, another point of the hyperbola.

151. In Fig. 93, were a tangent at B drawn to arc $A\,H\,B$, it would meet the axis in a point which, like all points on the axis, corresponds to itself. From that point the projection of that tangent on the lower plane would be parallel to $S\,B$, since they are to meet at infinity. Or if $S\,J$ is parallel to the tangent at B then J will be the projection of J' at infinity, where $S\,J$ meets the tangent; J will be therefore one point of the projection of said tangent on the lower plane; while another point would be, as previously stated, that in which the tangent at B meets the axis.

152. Analogously in Fig. 94, the tangents at M and N meet the axis, as at F and E; but the projectors $O\,M$ and $O\,N$ go to points of tangency *at infinity*; M and N are on a "vanishing line"; hence $O\,M$ is parallel to the tangent at infinity, that is to the *asymptote* (see Art. 134) through F; while the other asymptote is a parallel through E to $O\,N$.

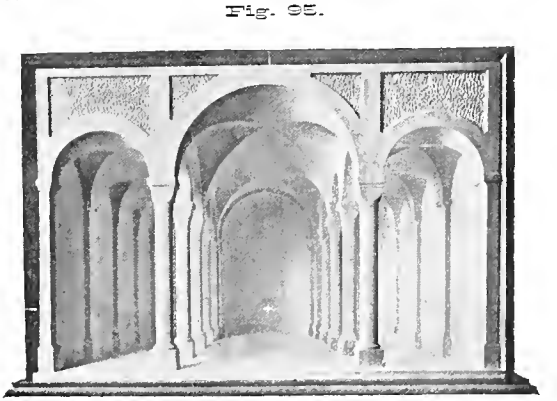
153. As in Fig. 93 the projectors from S to all points of the arc above the level of S could cut the lower plane only by being produced to the right, giving the right-hand branch of the hyperbola; so, in Fig. 94, the arc $M\,H\,N$, above the vanishing line, gives the lower branch of the hyperbola. To get a point of the latter, as h, and having already obtained any point x of the other branch, join H with N (the original of x) and get its intersection, g, with the axis. Then $x\,g\,h$ corresponds to $g\,N\,H$, and the ray $O\,H$ meets it at h, the projection of H.

The cases should be worked out in which the vanishing line is tangent to the circle or exterior to it.

154. The homological figures with which we have been dealing were *plane* figures. But it is possible to have space figures homological with each other.

In homological space figures corresponding lines meet at the same point *in a plane*, instead of the same point on a line. A vanishing *plane* takes the place of a vanishing *line*. The figure that is in homology with the original figure is called the *relief-perspective* of the latter. (See Art. 11.)

Fig. 95.



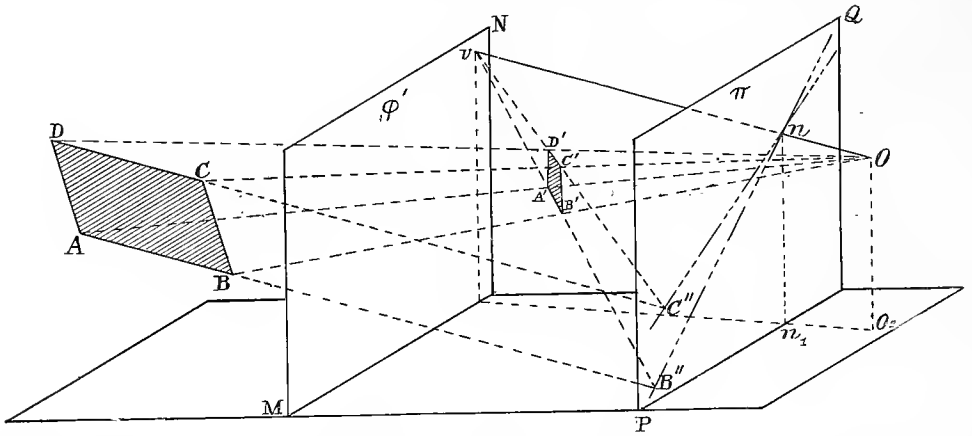
Remarkably beautiful effects can be obtained by the construction of homological space figures, as a glance at Fig. 95 will show. The figure represents a triple row of groined arches and is from a photograph of a model designed by Prof. L. Burmester.

Although not always requiring the use of the irregular curve and therefore not strictly the material for a topic in this chapter, its close analogy to the foregoing matter may justify a few words at this point on the construction of a relief-perspective.

155. In Fig. 96 the plane $P\,Q$ is called the *plane of homology* or *picture-plane*, and—adopting Cremona's notation—we will denote it by π. The *vanishing plane*, $M\,N$, or ϕ', is parallel to it. O is

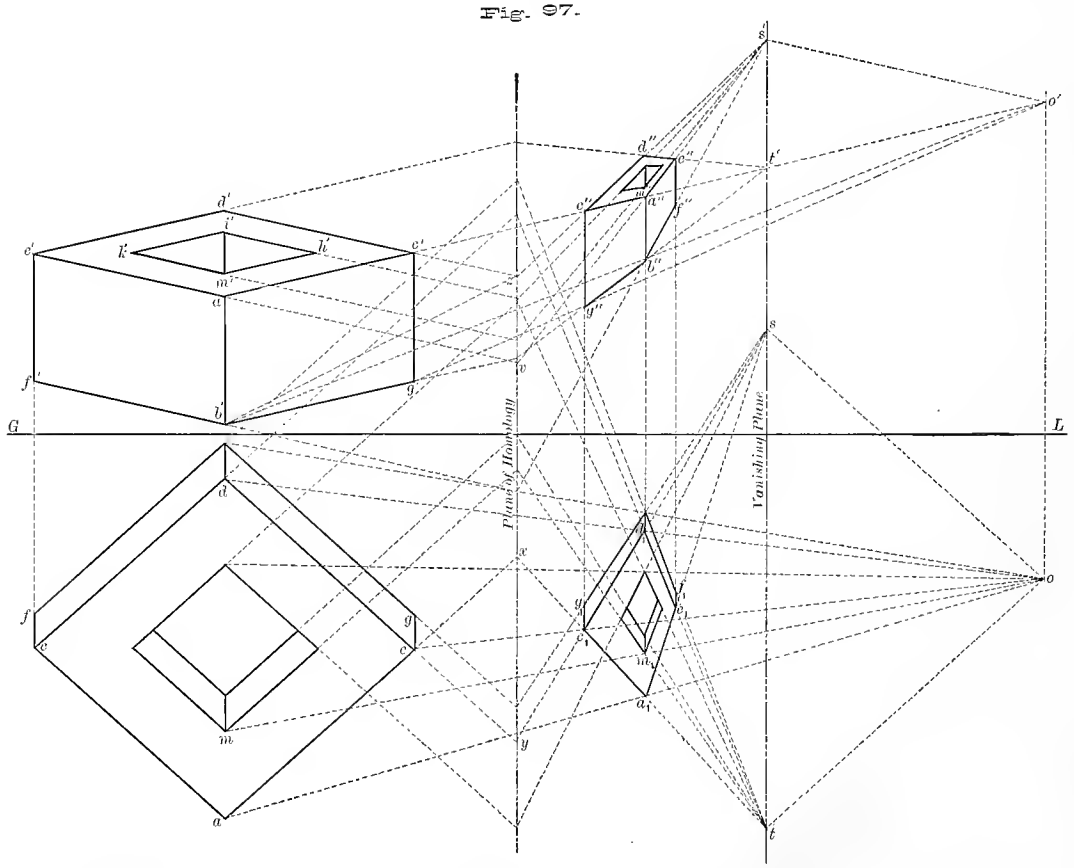
the *centre of homology* or *perspective-centre*. All points in the plane π are their own perspectives or, in other words, *correspond to themselves*. Therefore B'' is one point of the projection or perspective of the line $A\,B$, being the intersection of $A\,B$ with π. The line $O\,v$, parallel to $A\,B$, would meet the

Fig. 96.



latter *at infinity;* therefore v, in the vanishing plane, ϕ', would be the projection upon it of the point at infinity. Joining v with B'' and cutting $v\,B''$ by rays $O\,A$ and $O\,B$ gives $A'\,B'$ as the relief-perspective of $A\,B$. The plane through O and $A\,B$ cuts π in $B''\,n$, which is an *axis of homology* for $A\,B$ and $A'\,B'$, exactly as $m\,n$ in Fig. 92 is for $A_1\,B_1$ and $A_2\,B_2$.

Fig. 97.



As $D\,C$ is parallel to $A\,B$ (Fig. 96) a parallel to it through O is again the line $O\,v$.

The trace of $D\,C$ on π is C''. Joining v with C'' and cutting $v\,C''$ by rays $O\,D$, $O\,C$, obtains $D'\,C'$ in the same manner as $A'\,B'$ was derived. The originals of $A'\,B'$ and $C'\,D'$ are parallel lines; but we see that their relief-perspectives meet at v. The vanishing plane is therefore the locus* of the vanishing points of lines that are parallel on the original object; while the plane of homology is the locus of the axes of homology of corresponding lines; or, differently stated, any line and its relief-perspective will, if produced, meet on the plane of homology.

156. Fig. 97 is inserted here for the sake of completeness, although its study may be reserved, if necessary, until the chapter on projections has been read. In it a solid object is represented at the left in the usual views, plan and elevation; $G\,L$ being the *ground line* or axis of intersection of the planes on which the views are made. The planes π and ϕ' are interchanged, as compared with their positions in Fig. 96, and they are seen as lines, being assumed as perpendicular to the paper. The relief-perspective appears between them, in plan and elevation.

The lettering of $A\,B$ and $D\,C$, and the lines employed in getting their relief-perspectives, being identical with the same constructions in Fig. 96 ought to make the matter clear at a glance to all who have mastered what has preceded.

Burmester's *Grundzüge der Relief-Perspective* and Wiener's *Darstellende Geometrie* are valuable reference works on this topic for those wishing to pursue its study further; but for special work in the line of homological *plane* figures the student is recommended to read Cremona's *Projective Geometry* and Graham's *Geometry of Position*, the latter of which is especially valuable to the engineer or architect since it illustrates more fully the practical application of central projection to Graphical Statics.

LINK-MOTION CURVES.

157. *Kinematics* is the science which treats of pure motion, *regardless of the cause or the results of the motion.*

It is a purely kinematic problem if we lay out on the drawing-board the path of a point on the connecting-rod of a locomotive, or of a point on the piston of an oscillating cylinder, or of any point on one of the moving pieces of a mechanism. Such problems often arise in machine design, especially in the invention or modification of valve-motions.

Some of the motion-curves or point-paths that are discovered by a study of relative motion are without special name. Others, whose mathematical properties had already been investigated and the curves dignified with names, it was later found could be mechanically traced. Among these the most familiar examples are the Ellipse and the Lemniscate, the latter of which is employed here to illustrate the general problem.

The moving pieces in a mechanism are rigid and inextensible and are always under certain *conditions of restraint.* "Conditions of restraint" may be illustrated by the familiar case of the connecting-rod of the locomotive, one end of which is always attached to the driving-wheel at the crank-pin and is therefore constrained to describe a circle about the axle of that wheel, while the other end of the rod must move in a straight line, being fastened by the "wrist-pin" to the "cross-head," which slides between straight "guides." The first step in tracing a point-path of any mechanism is therefore the determination of the *fixed* points and a general analysis of the motion.

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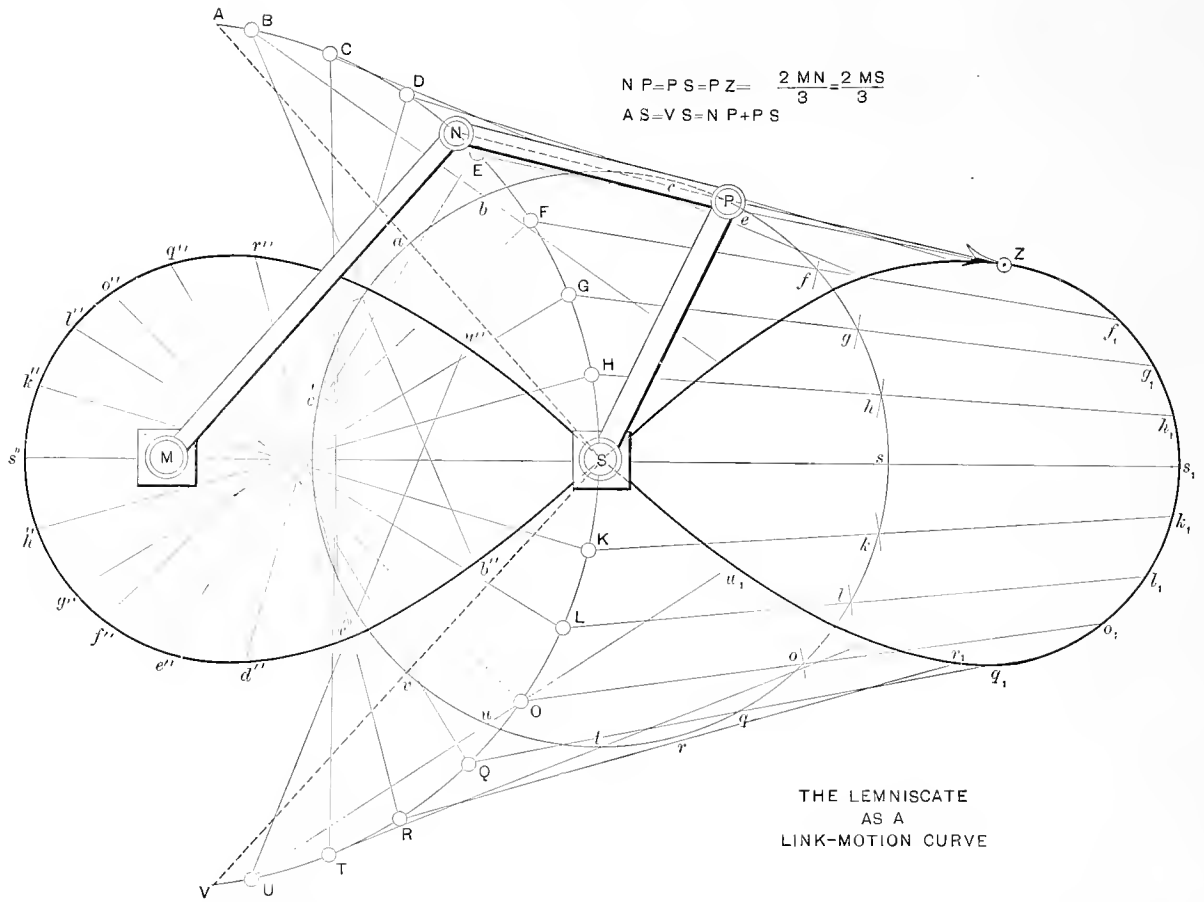
* *Locus* is the Latin for *place*; and in rather untechnical language, although in the exact sense in which it is used mathematically, we may say that the *locus* of points or lines is the place where you may expect to find them under their conditions of restriction. For example, the surface of a sphere is the *locus* of all points equidistant from a fixed point (its centre). The *locus* of a point moving in a plane so as to remain at a constant distance from a given fixed point, is a circle having the latter point as its centre.

158. We have given, in Fig. 98, two links or bars, MN and SP, fastened at N and P by pivots to a third link, NP, while their other extremities are pivoted on *stationary* axes at M and S. The only movement possible to the point N is therefore in a circle about M; while P is equally limited to circular motion about S. The points on the link NP, with the exception of its

Fig. 98.



extremities, have a compound motion, in curves whose form it is not easy to predict and which differ most curiously from each other. The figure-of-eight curve shown, otherwise the "Lemniscate of Bernoulli," is the point-path of Z, the link NP being supposed prolonged by an amount, PZ, equal to NP. Since NP is constant in length, if N were moved along to E the point P would have to be at a distance NP from E and also on the circle to which it is confined; therefore its new position, f, is at the intersection of the circle Psr by an arc of radius PN, centre F. Then Ff prolonged by an amount equal to itself gives f_1, another point of the Lemniscate, and to which Z has then moved. All other positions are similarly found.

If the motion of N is toward D it will soon reach a limit, A, to its further movement in that direction, arriving there at the instant that P reaches a, when NP and PS will be in one straight line, SA. In this position any movement of P either side of a will drag N back over its former path; and unless P moves to the left, past a, it would also retrace its path. P reaches a similar "dead point" at r.

To get the curve illustrated, the links NP and PS had to be equal, as also the distance MS to MN. By varying the proportions of the links, the point-paths would be correspondingly affected.

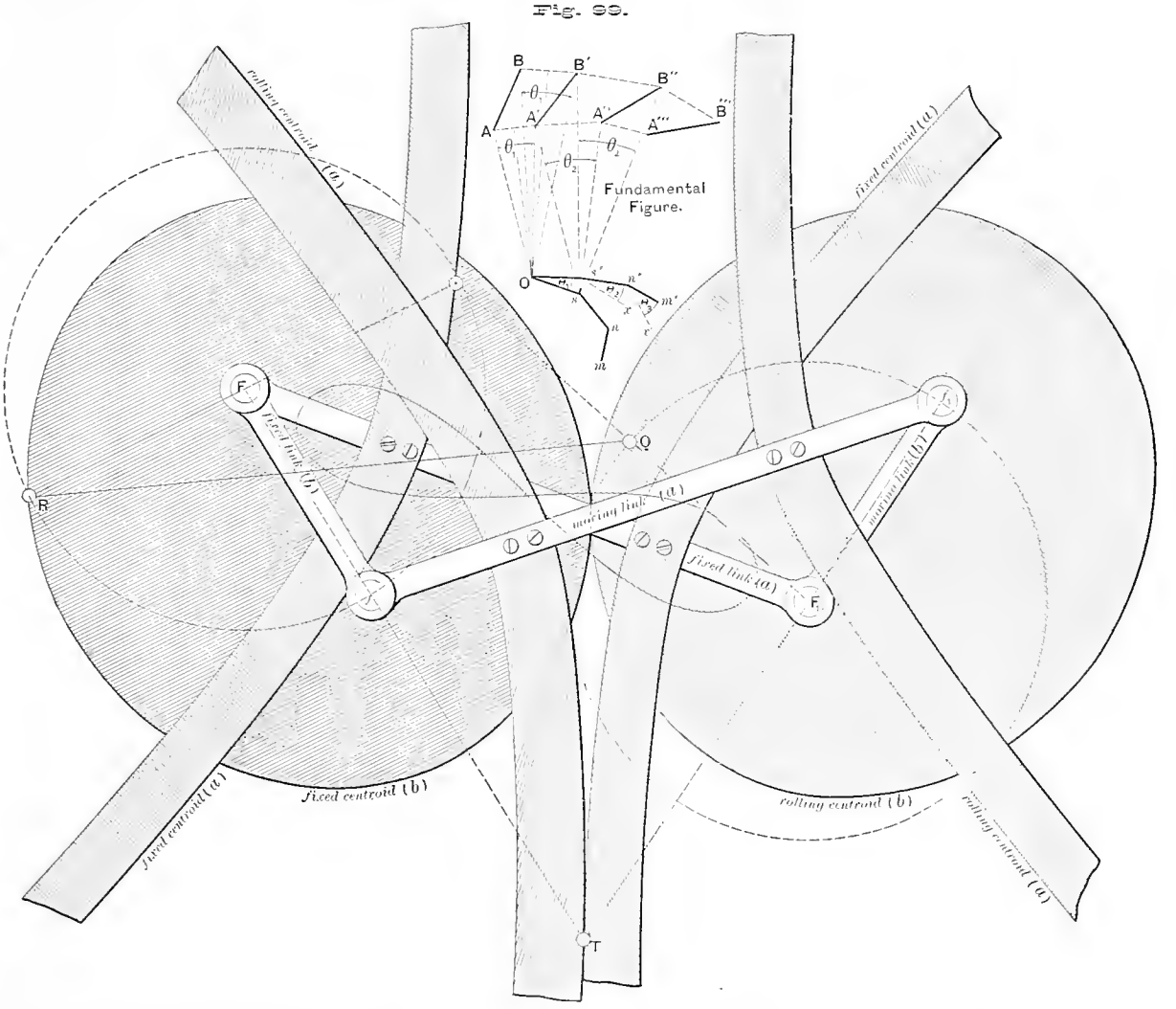
By tracing the path of a point on PN produced, and as far from N as Z is from P, the student will obtain an interesting contrast to the Lemniscate.

If M and S were joined by a link and the latter held rigidly in position, it would have been called the *fixed link*; and although its use would not have altered the motions illustrated and it is not essential that it should be drawn, yet in considering a mechanism, as a whole, the line joining the fixed centres always exists, in the imagination, as a link of the complete system.

INSTANTANEOUS CENTRES.—CENTROIDS.

159. Let us imagine a boy about to hurl a stone from a sling. Just before he releases it he runs forward a few steps, as if to add a little extra impetus to the stone. While taking those few steps a peculiar shadow is cast on the road by the end of the sling, if the day is bright. The

Fig. 99.



boy moves with respect to the earth; his hand moves in relation to himself, and the end of the sling describes a *circle* about his hand. The last is the only definite element of the three, yet it is sufficient to simplify otherwise difficult constructions relating to the complex curve which is described relatively to the earth.

A tangent and a normal to a *circle* are easily obtained, the former being, as need hardly be stated at this point, perpendicular to the radius at the point of tangency, while the normal simply coincides in direction with such radius. If the stone were released at any instant it would fly off in a straight line tangent to the circle it was describing about the hand as a centre; but such line would, at the instant of release, be tangent also to the compound curve. If then we wish a tangent at a given point of any curve generated by a point in motion, we have but to reduce that motion to circular motion about some moving centre; then joining the point of desired tangency with the—at that instant—position of the moving centre, we have the *normal*, a perpendicular to which gives the tangent desired.

A centre which is thus used *for an instant only* is called an *instantaneous centre.*

160. In Fig. 99 a series of instantaneous centres are shown and an important as well as interesting fact illustrated, viz., that every moving piece in a mechanism might be rigidly attached to a certain curve and by the rolling of the latter upon another curve, the link might be brought into all the positions which its visible modes of restraint compel it to take.

161. In the "Fundamental" part of the figure $A\,B$ is assumed to be one position of a link. We next find it, let us suppose, at $A'\,B'$, A having moved over $A\,A'$ and B over $B\,B'$. Bisecting $A\,A'$ and $B\,B'$ by perpendiculars intersecting at O, and drawing $O\,A$, $O\,A'$, $O\,B$, and $O\,B'$, we have $A\,O\,A' = \theta_1 = B\,O\,B'$, and O evidently a point about which, as a centre, the turning of $A\,B$ through the angle θ_1 would have brought it to $A'\,B'$. Similarly, if the next position in which we find $A\,B$ is $A''\,B''$, we may find a point s as the centre about which it might have turned to bring it there; the angle being θ_2, probably different from θ_1.

The points n and m are analogous to O and s.

If $O\,s'$ be drawn equal to $O\,s$ and making with the latter an angle θ_1, equal to the angle $A\,O\,A'$, and if $O\,s$ were rigidly attached to $A\,B$ the latter would be brought over to $A'\,B'$ by bringing $O\,s'$ into coincidence with $O\,s$. In the same manner, if we bring $s'\,n'$ upon $s\,n$ through an angle θ_2 about s, then the next position, $A''\,B''$, would be reached by $A\,B$. $O'\,s'\,n'\,m'$ is then part of a polygon whose rolling upon $O\,s\,n\,m$ would bring $A\,B$ into all the positions shown, provided the polygon and the line were so attached as to move as one piece. Polygons whose vertices are thus obtained are called *central polygons.*

If *consecutive* centres were joined we would have curves, called *centroids*, instead of polygons; the one corresponding to $O\,s\,n\,m$ being called the *fixed*, the other the *rolling* centroid. The perpendicular from O upon $A\,A'$ is a normal to that path. But if A were to move in a circular arc the normal to its path at any instant would be simply the radius to the position of A at that instant.

If then both A and B were moving in circular paths we would find the instantaneous centre at the intersection of the normals (radii) to the points A and B.

162. In Fig. 98 the instantaneous centre about which the whole link $N\,P$ is turning is at the intersection of radii $M\,N$ and $S\,P$ (produced); and calling it X we would have $X\,Z$ normal at Z to the Lemniscate.

163. The shaded portions of Fig. 99 illustrate some of the forms of centroids.

The mechanism is of four links, opposite links equal. Unlike the usual quadrilateral fulfilling this condition, the long sides cross, hence the name "anti-parallelogram."

The "fixed link (a)" corresponds to $M\,S$ of Fig. 98 and its extremities are the centres of rotation of the short links, whose ends, f and f_1, describe the dotted circles.

For the given position T is evidently the instantaneous centre. Were a bar pivoted at T and

fastened at right angles to "moving link (a)," an *infinitesimal* turning about T would move "link (a)" exactly as under the old conditions.

By taking "link (a)" in all possible positions and, for each, prolonging the radii through its extremities, the points of the fixed centroid are determined. Inverting the combination so that "moving link (a)" and its opposite are interchanged, and proceeding as before, gives the points of "rolling centroid (a)."

These centroids are branches of hyperbolas having the extremities of the long links as foci.

By holding a short link stationary, as "fixed link (b)," an elliptical fixed centroid results; "rolling centroid (b)" being obtained, as before, by inversion. The foci are again the extremities of the fixed and moving links.

Obviously the curved pieces represented as screwed to the links would not be employed in a practical construction, and they are only introduced to give a more realistic effect to the figure and possibly thereby conduce to a clearer understanding of the subject.

164. It is interesting to notice that the Lemniscate occurs here under new conditions, being traced by the middle point of "moving link (a)."

The study of kinematics is both fascinating and profitable, and it is hoped that this brief glance at the subject may create a desire on the part of the student to pursue it further in such works as Reuleaux' *Kinematics of Machinery* and Burmester's *Lehrbuch der Kinematik*.

Before leaving this topic the important fact should be stated, which now needs no argument to establish, that the instantaneous centre for any position of the moving piece, is the *point of contact* of the rolling and fixed centroids.

165. We shall have occasion to use this principle in drawing tangents and normals to the

TROCHOIDS

which are the principal *Roulettes*, or *roll-traced curves*, and which may be defined as follows:—

If in the same plane one of two circles rolls upon the other without sliding, the path of any point on the radius of the rolling circle or on the radius produced is a *trochoid*.

166. *The Cycloid.* Since a straight line may be considered a circle of *infinite radius* the above definition would include the curve traced by a point on the circumference of a locomotive wheel as it rolls along the rail, or of a carriage wheel on the road. This curve is known as a *cycloid** and is shown in $T\ n\ a\ b\ c$, Fig. 100. It is the proper outline for a portion of each tooth in a certain case of gearing, viz., where one wheel has an infinite radius, that is, becomes a "rack." Were T_6 a ceiling-corner of a room, and T_{12} the diagonally opposite floor-corner, a weight would slide from T_6 to T_{12} more quickly on guides curved in cycloidal shape than if shaped to any other curve or if straight. If started at c or any other point of the curve it would reach T_{12} as soon as if started at T_6.

167. In beginning the construction of the cycloid we notice first that as $T\ V\ D$ rolls on the straight line $A\ B$ the arrow $D\ R\ T$ will be reversed in position (as at $D_5\ T_6$) as soon as the semi-circumference $T\ 3\ D$ has had rolling contact with $A\ B$. The *tracing point* will then be at T_6, its maximum distance from $A\ B$.

When the wheel has rolled itself out once upon the rail the point T will again come in contact with the rail, as at T_{12}.

* "Although the invention of the cycloid is attributed to Galileo, it is certain that the family of curves to which it belongs had been known and some of the properties of such curves investigated, nearly two thousand years before Galileo's time, if not earlier. For ancient astronomers explained the motion of the planets by supposing that each planet travels uniformly round a circle whose centre travels uniformly around another circle."—Proctor, *Geometry of Cycloids.*

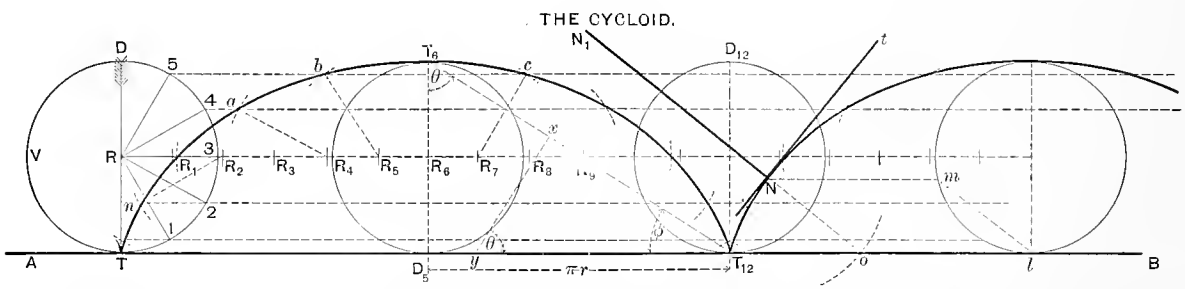
The distance $T T_{12}$ evidently equals $2 \pi r$, when $r = T R$.

If the semi-circumference $T 3 D$ (equal to πr) be divided into any number of equal parts; and also the path of centres $R R_6$ (again $= \pi r$) into the same number of equal parts, then as the points 1, 2, etc., come in contact with the rail, the centre R will take the positions R_1, R_2, etc., directly above the corresponding points of contact. A sufficient rolling of the wheel to bring point 2 upon $A B$ would evidently raise T from its original position to the former level of 2. But as T must always be at a radius' distance from R, and the latter would by that time be at R_2, we would find T located at the intersection (n) of the dotted *line of level* through 2 by an arc of radius $R T$, centre R_2. Similarly for other points.

The construction, summarized, involves the drawing of *lines of level* through *equidistant* points of division on a semi-circumference of the rolling circle, and their intersection by arcs of constant radius (that of the rolling circle) from centres which are the successive positions taken by the centre of the rolling circle.

It is worth while calling attention to a point occasionally overlooked by the novice, although almost self-evident, that in the position illustrated in the figure the point T drags behind the centre R until the latter reaches R_6, when it passes and goes ahead of it. From R_7 the line of level through 5 could be cut not alone at c by an arc of radius $c R_7$ but also in a second point; evidently but one of these points belongs to the cycloid, and the choice depends upon the direction of turning and the relative position of the rolling centre and the moving point. This matter requires more thought in drawing trochoidal curves in which both circles have finite radii, as will appear later.

Fig. 100.



168. Were points T_6 and T_{12} given, and the semi-cycloid $T_6 T_{12}$ desired, we can readily ascertain the "base," $A B$, and generating circle, as follows:—Join T_6 with T_{12}; at any point of such line, as x, erect a perpendicular, $x y$; from the similar triangles $x y T_{12}$ and $T_6 D_5 T_{12}$ having angle ϕ common and angles θ equal we see that

$$x y : x T_{12} :: T_6 D_5 : D_5 T_{12} :: 2 r : \pi r :: 2 : \pi :: 1 : \frac{\pi}{2} \text{ or, very nearly, as } 14 : 22.$$

If, then, we lay off $x T_{12}$ equal to *twenty-two* equal parts on any scale, and a perpendicular, $x y$, *fourteen* parts of the same scale, the line $y T_{12}$ will be the base of the desired curve; while the diameter of the generating circle will be the perpendicular from T_6 to $y T_{12}$ prolonged.

169. To draw a tangent to a cycloid at any point is a simple matter, if we see the analogy between the *point of contact* of the wheel and rail at any instant, and the *hand* used in the former illustration (Art. 159). At any one moment each point on the entire wheel may be considered as describing an infinitesimal arc of a circle whose radius is the line joining the point with the point of contact on the rail. The tangent at N, for example, (Fig. 100), would be $t N$, perpendicular to the normal, $N o$, joining N with o; the latter point being found by using N as a centre and

cutting $A\,B$ by an arc of radius equal to $m\,l$, in which m is a point *at the level* of N on *any* position of the rolling circle, while l is the corresponding point of contact. The point o might also have been located by the following method: Cut the line of centres by an arc, centre N, radius $T\,R$; o would obviously be vertically below the position of the rolling centre thus determined.

170. *The Companion to the Cycloid.* The kinematic method of drawing tangents, just applied, was devised by Roberval, as also the curve named by him the "Companion to the Cycloid," to which allusion has already been made (Art. 120) and which was invented by him in 1634 for the purpose of solving a problem upon which he had spent six years without success and which had foiled Galileo, viz., the calculating of the area between a cycloid and its base. Galileo was reduced to the expedient of comparing the area of the cycloid with that of the rolling circle by weighing paper models of the two figures. He concluded that the area in question was nearly but not exactly three times that of the rolling circle. That the latter would have been the correct solution may be readily shown by means of the "Companion," as will be found demonstrated in Art. 172.

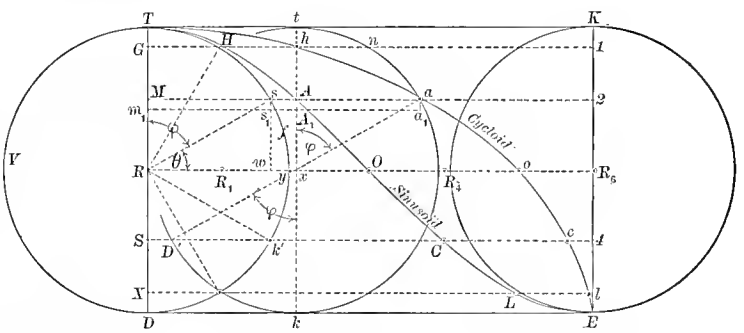
171. Suppose two points coincident at T, (Fig. 101), and starting simultaneously to generate curves, the first of these points to trace the cycloid during the rolling of circle $T\,V\,D$ while the second is to move independently of the circle and so as to be always *at the level* of the point tracing the cycloid, yet at the same time *vertically above* the point of contact of the circle and base. This makes the second point always as far from the initial vertical diameter, or *axis*, of the cycloid as the length of the arc from T to whatever level the tracing point of the latter has then reached; that is, $M\,A$ equals arc $T\,H\,s$; $R\,O$ equals quadrant $T\,s\,y$.

Adopting the method of Analytical Geometry, by using O as a reference point, or *origin*, we may reach any point, A, on the curve, by *co-ordinates*, as $O\,x$, $x\,A$, of which the horizontal is called an *abscissa*, the vertical an *ordinate*. By the preceding construction $O\,x$ equals *arc* $s\,f\,y$, while $x\,A$ equals $s\,w$—the *sine* of the same arc. The "Companion to the Cycloid" is therefore a *curve of sines* or *sinusoid*, since starting from O the abscissas are equal to or proportional to the arc of a circle while the ordinates are the sines of those arcs.

This curve is particularly interesting as "expressing the law of the vibration of perfectly elastic solids; of the vibratory movement of a particle acted upon by a force which varies directly as the distance from the origin; approximately, the vibratory movement of a pendulum; and exactly the law of vibration of the so-called mathematical pendulum."*

172. From the symmetry of the sinusoid with respect to $R\,R_6$ and to O we have *area* $T\,A\,O\,R = E\,C\,O\,R_6$; adding *area* $D\,E\,L\,O\,R$ to both members we have the area between the sinusoid and $T\,D$ and $D\,E$ equal to the rectangle $R\,E$, or one-half the rectangle $D\,E\,K\,T$; or to $\frac{1}{2}\,\pi\,r \times 2\,r = \pi\,r^2$, the area of the rolling circle.

Fig. 101.



As $T\,A\,C\,E$ is but half of the entire sinusoid it is evident that the total area below the curve is twice that of the generating circle.

The area between the cycloid and its "companion" remains to be determined, but is readily ascertained by noting that as any point of the latter, as A, is on the vertical diameter of the circle

* Wood. *Elements of Co-ordinate Geometry*, p. 209.

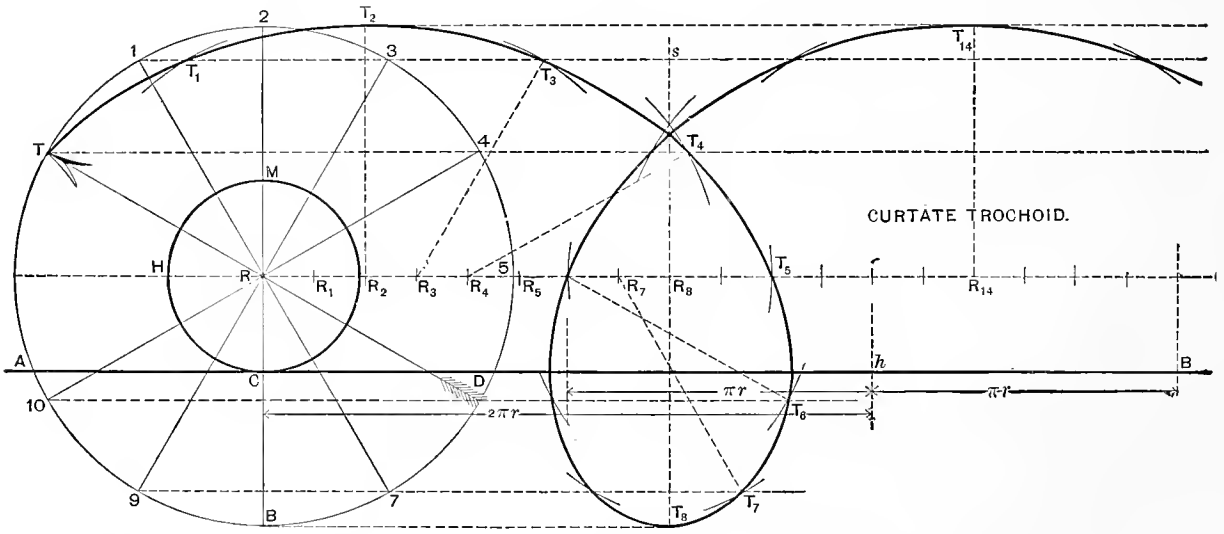
passing through the then position of the tracing point, as a, the distance, $A\,a$, between the two curves at any level, is merely the semi-chord of the rolling circle at that level. But this, evidently, equals $M\,s$, the semi-chord at the same level on the equal circle. The equality of $M\,s$ and $A\,a$ makes the elementary rectangles $M\,s\,s_1\,m_1$ and $A\,A_1\,a_1\,a$ equal; and considering all the possible similarly constructed rectangles of infinitesimal altitude, the sum of those on semi-chords of the rolling circle would equal the area of the semi-circle $T\,D\,y$, which is therefore the extent of the area between the two curves under consideration.

The figure showing but half of a cycloid, the total area between it and its "companion" must be that of the rolling circle. Adding this to the area between the "companion" and the base makes the total area between cycloid and base equal to *three* times that of the rolling circle.

173. The paths of points carried by and in the plane of the rolling circle, though not on its circumference, are obtained in a manner closely analogous to that employed for the cycloid.

In Fig. 102 the looped curve, traced by the arrow-point while the circle $C\,H\,M$ rolls on the base $A\,B$, is called the *Curtate Trochoid.* To obtain the various positions of the tracing point T describe a circle through it from centre R. On this circle lay off any even number of equal arcs and draw radii from R to the points of division; also "lines of level" through the latter. The radii drawn intercept equal arcs on the rolling circle $C\,H\,M$. While the first of these arcs rolls upon $A\,B$ the point T turns through the angle $T\,R\,1$ about R and reaches the line of level through point *1*. But T is always at the distance $R\,T$ (called the *tracing radius*) from R; and, as R has reached R_1 in the rolling supposed, we find T_1—the new position of T—by an arc from R_1, radius $T\,R$, cutting said line of level.

Fig. 102.



After what has preceded, the figure may be assumed to be self-interpreting, each position of T being joined with the position of R which determined it.

174. Were a tangent wanted at any point, as T_7, we have, as before, to determine the point of contact of rolling circle and line when T reached T_7, and use it as an instantaneous centre. T_7 was obtained from R_7; and the point of contact must have been vertically below the latter and on $A\,B$. Joining such point to T_7 gives the normal, from which the tangent follows in the usual way.

175. *The Prolate Trochoid.* Had we taken a point *inside* of the circle $C\,H\,M$ and constructed its path the only difference between it and the curve illustrated would have been in the *name* and the

shape of the curve. An undulating, wavy path would have resulted, called the *prolate trochoid;* but, as before, we would have described a circle through the tracing point; divided it into equal parts; drawn lines of level, and cut them by arcs of constant radius, using as centres the successive positions of *R*.

HYPO-, EPI- AND PERI-TROCHOIDS.

176. Circles of *finite* radius can evidently be tangent in but two ways—either *externally*, or *internally;* if the latter, the larger may roll on the one within it, or the smaller may roll inside the larger. When a small circle rolls within a larger the *radius* of the latter may be greater than the *diameter* of the rolling circle, or may equal it, or be smaller. On account of an interesting property of the curves traced by points in the planes of such rolling circles, viz., their capability of being generated, trochoidally, in two ways, a nomenclature was necessary which should indicate how each curve was obtained. This is included in the tabular arrangement of names below and which was the outcome of an investigation made by the writer in 1887 and presented before the American Association for the Advancement of Science.* In accepting the new terms advanced at that time Prof. Francis Reuleaux suggested the names *Ortho-cycloids* and *Cyclo-orthoids* for the classes of curves of which the cycloid and involute are respectively representative; *orthoids* being the paths of points in a fixed position with respect to a straight line rolling upon *any* curve, and *cyclo-orthoid* therefore implying a circular director or base-curve. These appropriate terms have been incorporated in the table.

For the last column a point is considered as *within* the rolling circle of infinite radius when on the normal to its initial position and on the side toward the centre of the fixed circle.

As will be seen by reference to the Appendix, the curves whose names are preceded by the same letter may be identical. Hence the terms *curtate* and *prolate*, while indicating whether the tracing point is beyond or within the circumference of the rolling circle, give no hint as to the actual *form* of the curves.

In the table *R* represents the radius of the rolling circle, *F* that of the fixed circle.

NOMENCLATURE OF TROCHOIDS.

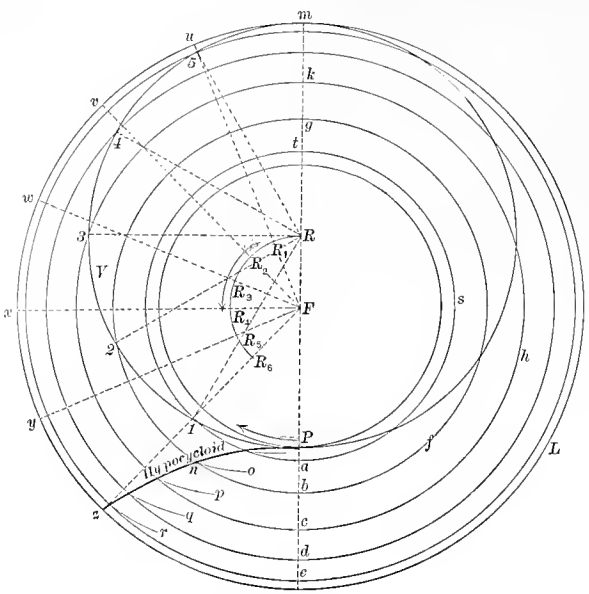
Position of Traeing or Deseribing Point.	Circle rolling upon Straight Line. $F = \infty$	Circle rolling upon circle.					Straight Line rolling upon Circle. $R = \infty$
		External contact.	Internal contact.				
			Larger Circle rolling.	Smaller circle rolling.			
				$2R > F.$	$2R < F.$	$2R = F.$	
	Ortho-cycloids.	Epitrochoids.	Peritroehoids.	Major Hypotroehoids.	Minor Hypotroehoids.	Medial Hypotroehoids.	Cyclo-orthoids.
On circumference of rolling eirele.	Cycloid.	(a) Epicycloid.	(a) Perieycloid	(d) Major Hypocycloids.	(d) Minor Hypocycloid.	Straight Hypocycloid.	Involute.
Within Cireumferenee.	Prolate Trochoid.	(b) Prolate Epitrochoid.	(e) Prolate Peritrochoid.	(e) Major Prolate Hypotrochoid.	(f) Minor Prolate Hypotroehoid.	(g) Prolate Elliptical Hypotrochoid.	Prolate Cyclo-orthoid.
Without Cireumferenee.	Curtate Trocboid.	(e) Curtate Epitrochoid.	(b) Curtate Peritrochoid.	(f) Major Curtate Hypotroehoid.	(e) Minor Curtate Hypotrochoid.	(g) Curtate Elliptical Hypotrochoid.	Curtate Cyclo-orthoid.

177. From the above we see that the prefix *epi* (*over* or *upon*) denotes the curves resulting from external contact; *hypo* (*under*) those of internal contact with smaller circle rolling; while *peri* (*about*) indicates the third possibility as to rolling.

* Re-printed in substance in the Appendix.

178. The construction of these curves is in closest analogy to that of the cycloid. If, for example, we desire a *major hypocycloid* we first draw two circles, $m\,V\,P$, $m\,x\,L$, (Fig. 103), tangent internally, of which the rolling circle has its diameter greater than the radius of the fixed circle. Then, as for the cycloid, if the *tracing-point* is P, we divide the semi-circumference $m\,V\,P$ into equal parts and from the *fixed centre*, F, describe circles through the points of division, as those through *1*, *2*, *3*, *4* and *5*. These replace the "lines of level" of the cycloid, and may be called *circles of distance*, as they show the distance from F of the point P, for definite amounts of angular rotation of the latter. For, if the circle $P\,V\,m$ were simply to rotate about R, the point P would reach m during a semi-rotation and would then be at its maximum distance from F. After turning through the equal arcs $P\,1$, *1-2*, etc., its distances from F would be $F\,a$ and $F\,b$ respectively.

Fig. 103.



If, however, the turning of P about R is due to the rolling of circle $P\,V\,m$ upon the arc $m\,x\,z$ then the *actual position* of P, for any amount of turning about R, is determined by noting the new position of R, due to such rolling, as R_1, R_2, etc., and from it as a centre cutting the proper circle of distance by an arc of radius $R\,P$.

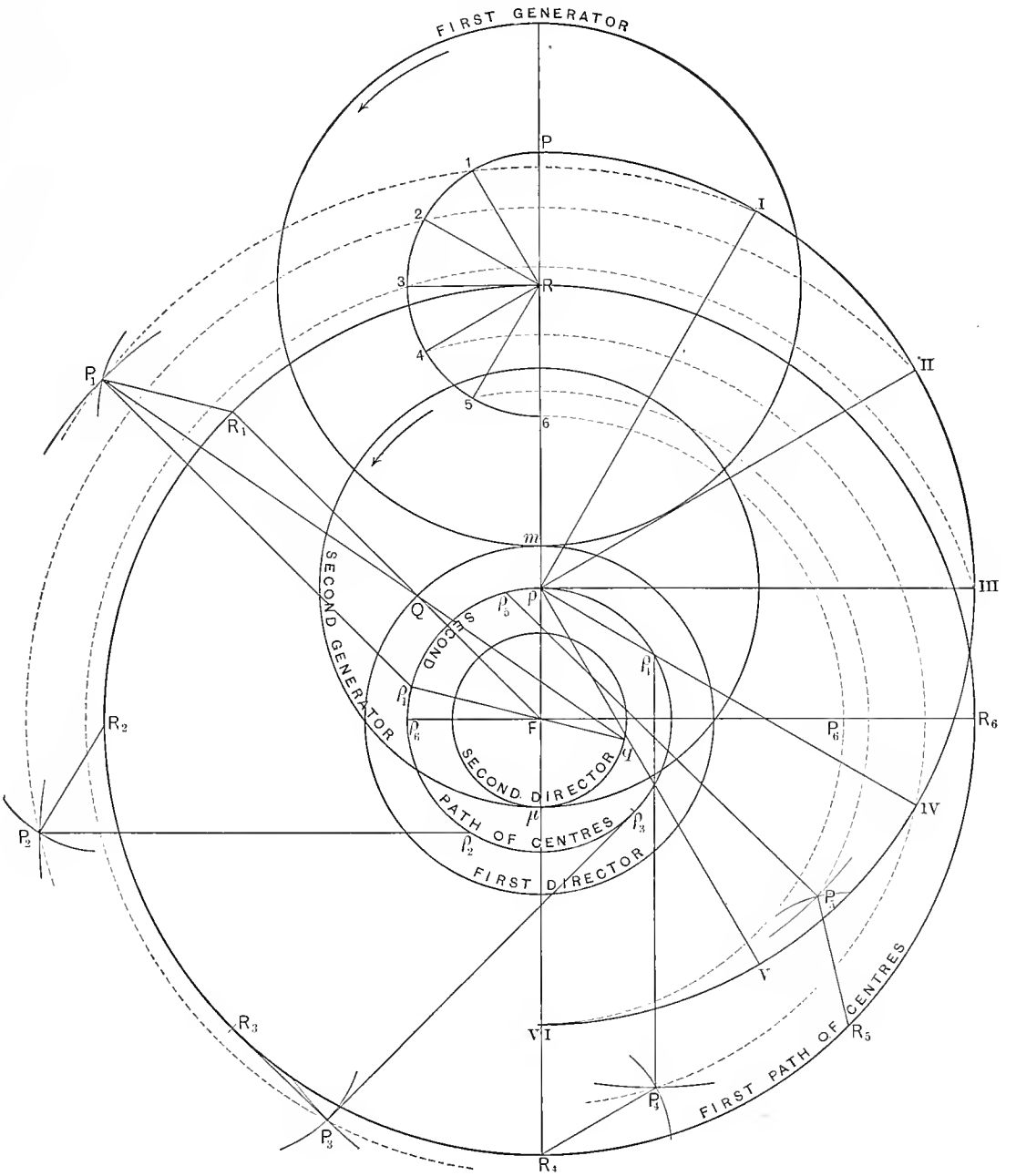
Since the radius of the smaller circle is in this case three-fourths that of the larger, the angle $m\,F\,z$ (135°) at the centre of the latter intercepts an arc, $m\,x\,z$, *equal to* the 180° arc, $m\,V\,P$, on the smaller circle. *Equal arcs on unequal circles are subtended by angles at the centre which are inversely proportional to the radii.*

While arc $m\,V\,P$ rolls upon arc $m\,x\,z$ the centre R will evidently move over circular arc $R\text{---}R_6$. Divide $m\,x\,z$ into as many equal parts as $m\,V\,P$ and draw radii from F to the points of division; these cut the path of centres at the successive positions of R. When arc $m\,5\text{-}4$, for example, has rolled upon its equal $m\,u\,v$ then R will have reached R_2; P will have turned about R through angle $P\,R\,2 = m\,R\,4$ and will be at n, the intersection of $b\,f\,g$—the circle of distance through *2*—by an arc, centre R_2, radius $R\,P$. Similarly for other points.

179. To trace the path of any point on the circumference of a circle so rolling as to give the *epi-* or *peri*-cycloid requires a construction similar at every step to that just illustrated. The same remark applies equally to the determination of the paths of points within or beyond the circumference of the rolling circle, as will be seen by reference to Fig. 104, in which the path of the point P is determined (a) as carried by the circle called "first generator," rolling on the exterior of the "first director"; (b) as carried by the "second generator" which rolls on the exterior of the "second director"—which it also encloses. In the first case the resulting curve is a *prolate epitrochoid;* in the second a *curtate peritrochoid.* Proceeding in the usual way, a semi-circle is drawn through P from each rolling centre, R and ρ. Dividing these semi-circles into the same number of equal parts draw next the dotted "circles of distance" through these points, all from centre F. The figure illustrates the special case where the larger set of "circles of distance" divides both semi-circumferences into equal parts. The successive positions of P, as P_1, P_2, etc., are then located by

arcs of radii $R\,P$ or $\rho\,P$, struck from the successive positions of R or ρ and intersecting the proper "circle of distance." For example, the turning of P through the angle $P\,R\,1$ about R would bring P somewhere upon the circle of distance through point *1;* but that amount of turning would be due to the rolling of the first generator over the arc $m\,Q$, which would carry R to R_1; P would

Fig. 104.



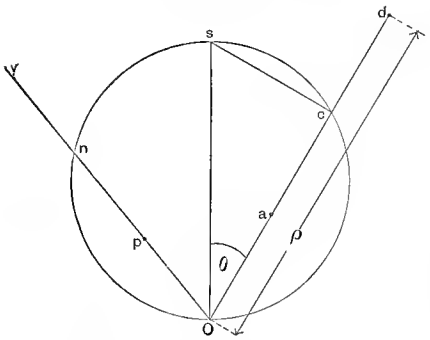
therefore be at P_1, at a distance $R\,P$ from R_1 and on the dotted arc through *1.* Similarly in relation to ρ. Each position of P is joined with each of the centres from which it could be obtained.

SPECIAL TROCHOIDS.

180. *The Ellipse and Straight Line.* Two circles are called Cardanic* if tangent *internally* and the diameter of one is twice that of the other. If the smaller roll in the larger all points in the plane of the generator will describe *ellipses* except points *on* the circumference, each of which will move in a straight line—a diameter of the director. Upon this latter property the mechanism known as "White's Parallel Motion" is based, in which a piston-rod or other piece intended to have reciprocating rectilinear motion is pivoted to a small gear-wheel or pinion which rolls on the interior of a toothed annular wheel of twice the diameter of the pinion.

181. *The Limaçon and Cardioid.* The Limaçon is a curve whose points may be obtained by drawing random secants through a point on the circumference of a circle and on each laying off a constant distance, on each side of the second point in which the secant cuts the circle.

Fig. 105.



In Fig. 105 let $O\,v$ and $O\,d$ be random secants of the circle $O\,n\,s$; then if $n\,v$, $n\,p$, $c\,a$ and $c\,d$ are each equal to some constant, b, we shall have v, p, a and d as four points of a Limaçon. Referring any point as d to O and the diameter $O\,s$, we have $O\,d = \rho = O\,c + c\,d = 2\,r\cos\theta + b$, while $O\,a = 2\,r\cos\theta - b$, whence the general polar equation for the Limaçon, $\rho = 2\,r\cos\theta + b$.

When $b = 2\,r$ the Limaçon becomes the heart-shaped curve called the *Cardioid.*†

182. All Limaçons, general and special, may be generated either as epi- or peri-trochoidal curves: as epi-trochoids the generator and director must have *equal* diameters, any point *on* the circumference of the generator then tracing a Cardioid, while any point on the radius (or radius produced) describes a Limaçon; as peri-trochoids the larger of a pair of Cardanic circles must roll on the smaller, the Cardioid and Limaçon then resulting, as before, from the motion of points respectively *on* the circumference of the generator or *within* or *without* it.

183. In Fig. 106 the Cardioid is obtained as an epicycloid, being traced by point P during one revolution of the generator $P\,H\,m$ about an equal directing circle $m\,s\,O$.

As a Limaçon we may get points of the Cardioid, as y and z, by drawing a secant through O and laying off $s\,y$ and $s\,z$ each equal to $2\,r$.

184. *The Limaçon as a Trisectrix.* Three famous problems of the ancients were the squaring of the circle, the duplication of the cube and the trisection of an angle. Among the interesting curves invented by early mathematicians for the purpose of solving the latter problem were the Quadratrix and Conchoid, whose construction is given later in this chapter; but it has been found that certain trochoids also possess this interesting property, among them the Limaçon of Fig. 106, frequently called the *Epitrochoidal Trisectrix.* Its construction as an epitrochoid need not be described in detail after what has preceded.

As a Limaçon we would find points as G and X by drawing from R a secant $R\,X$ to the circle called "path of centres" and making $S\,X$ and $S\,G$ each equal to the radius of that circle.

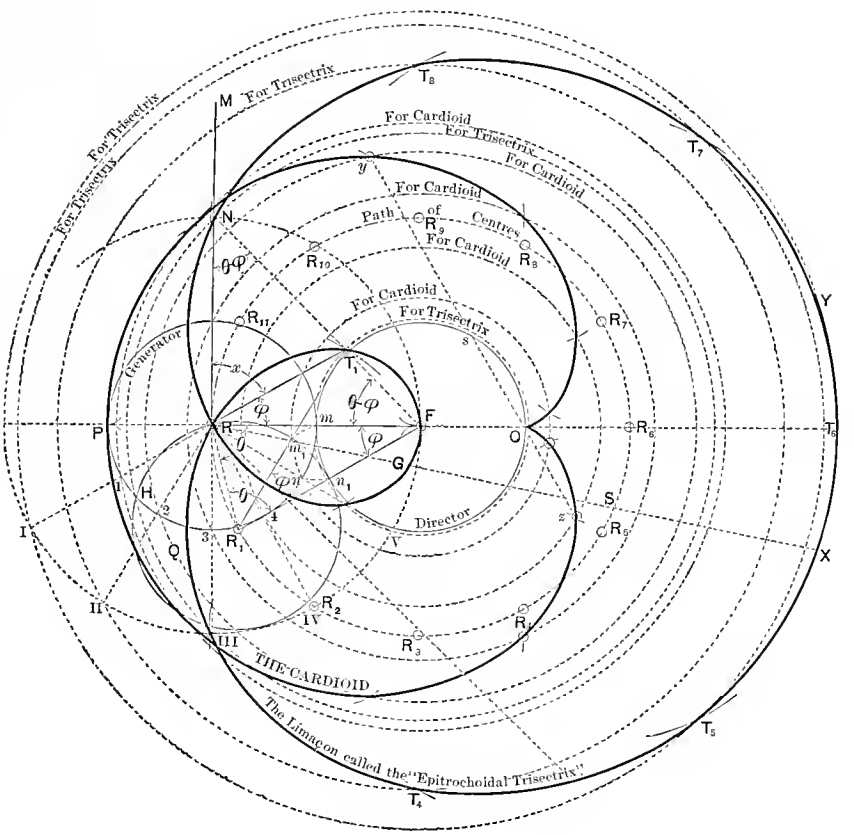
185. To trisect an angle, as $M\,R\,F$, bisect one side of the angle as $R\,F$ at m; use $m\,R$ and $m\,F$ as radii for generator and director respectively of an epitrochoid having a tracing radius, $R\,F$, equal to twice that of the generator. Make $R\,N = R\,F$ and draw $N\,F$; this will cut the Limaçon

* Term due to Reuleaux and based upon the fact that Cardano (16th century) was probably the first to investigate the paths described by points during their rolling.

† From *Cardis*, the Latin for *heart.*

$F\,T_1\,R\,Q$ (traced by point F as carried by the given generator) in a point T_1. The angle $T_1\,R\,F$ will then be one-third of $N\,R\,F$, which may be proved as follows: F reaches T_1 by the rolling of arc $m\,n$ on arc $m\,n_1$. These arcs are subtended by equal angles, ϕ, the circles being equal. During this rolling R reaches R_1, bringing $R\,F$ to $R_1\,T_1$. In the triangles $T_1\,R_1\,F$ and $R\,F\,R_1$ the side $F\,R_1$ is common, angles ϕ equal and side $R_1\,T_1$ equal to side $R\,F$; the line $R\,T_1$ is therefore parallel to $R_1\,F$, whence angle $T_1\,R\,F$ must also equal ϕ. In the triangle $R\,F\,R_1$ we denote by θ the angles opposite the equal sides $R\,F$ and $R_1\,F$; then $2\,\theta + \phi = 180°$ or $\theta = \frac{180° - \phi}{2}$. In triangle $N\,R\,F$ we have the angle at F equal to $\theta - \phi$, and $2\,(\theta - \phi) + x + \phi = 180°$ which gives $x = 2\,\phi$ by substituting value of θ from previous equation.

Fig. 106.



186. The *Involute.* As the opposite extreme of a circle rolling on a straight line we may have the latter rolling on a circle. In this case the *rolling* circle has an *infinite radius.* A point on the straight line describes a curve called the *involute.* This would be the path of the end of a thread if the latter were in tension while being unwound from a spool.

In Fig. 107 a rule is shown, tangent at u to a circle on which it is supposed to roll. Were a pencil-point inserted in the centre of the circle at j (which is on line $u\,x$ produced) it would trace the involute. When j reaches a the rule will have had rolling contact with the base circle over an arc $u\,t\,s\,\cdots\,a$ whose length equals line $u\,x\,j$. Were a the initial point we may obtain b, c, etc., by making *tangent* $m\,b$ = *arc* $m\,a$; *tangent* $n\,c$ = *arc* $n\,a$, etc. Each tangent thus equals the arc from initial point to point of tangency.

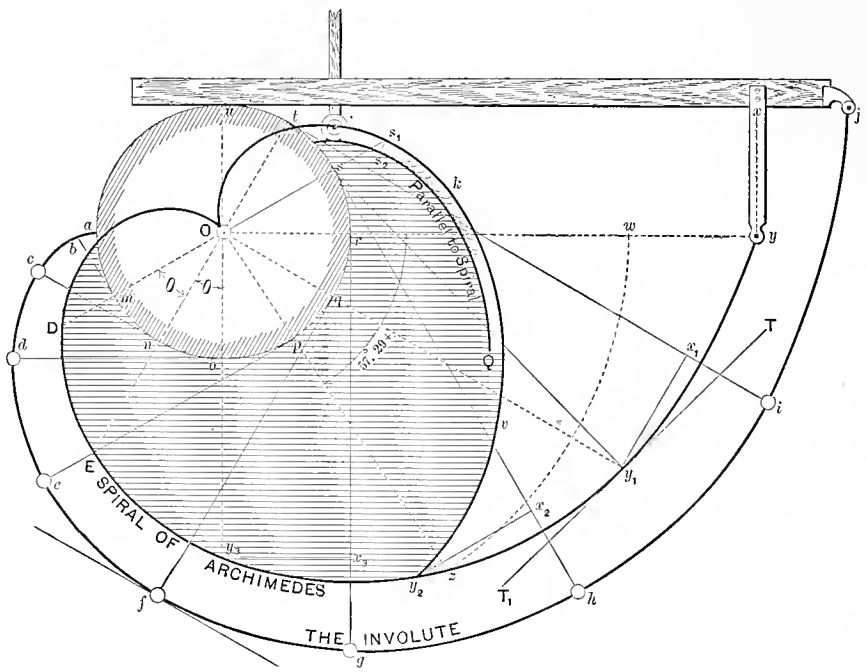
187. The circle from which the involute is derived or *evolved* is called the *evolute*. Were a hexagon or other figure to be taken as an evolute a corresponding involute could be derived; but the name "involute," unqualified, is understood to be that obtained from a circle.

From the law of formation of the involute the rolling line is in all its positions a normal to the curve; the point of tangency on the evolute is an instantaneous centre, and a tangent at any point as f is a perpendicular to the tangent, $f\,q$, from f to the base circle.

Like the cycloid, the involute is a correct working outline for the teeth of gear-wheels; and gears manufactured on the involute system are to a considerable degree supplanting other forms.

A surface known as the *developable helicoid* is formed by moving a line so as to be always *tangent* to a given helix. It is interesting in this connection to notice that any plane perpendicular to the axis of the helix would cut such a surface in a pair of involutes.*

Fig. 107.



188. *The Spiral of Archimedes.* This is the curve that would be generated by a point having a uniform motion around a fixed point—the *pole*—combined with uniform motion toward or from the pole.

In Fig. 107, with O as the pole, if the angles θ are equal and $O\,D$, $O\,E$ and $O\,y_3$ are in arithmetical progression then the points D, E and y_3 are points of an Archimedean Spiral.

This spiral can be trochoidally generated, simultaneously with the involute, by inserting a pencil point at y in a piece carried by—and at right angles with—the rule, the point y being at a distance, $x\,y$, from the contact-edge of the rule, equal to the radius $O\,s$ of the base circle of the involute; for

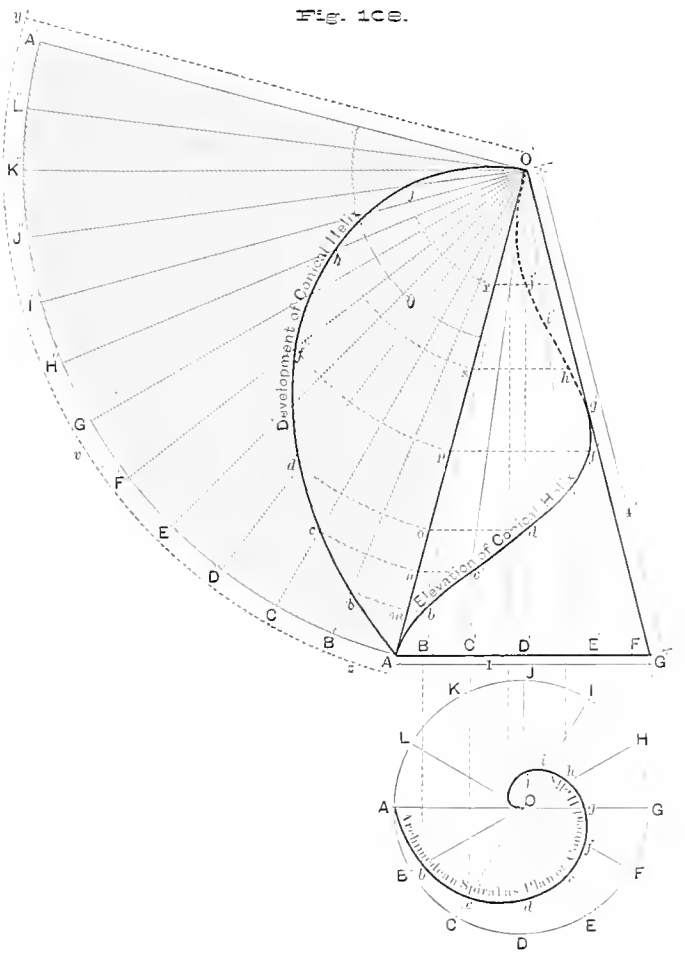
* The day of writing the above article the following item appeared in the New York *Evening Post:* "Visitors to the Royal Observatory, Greenwich, will hereafter miss the great cylindrical structure which has for a quarter century and more covered the largest telescope possessed by the Observatory. Notwithstanding its size the Astronomer Royal has now procured through the Lords Commissioners a telescope more than twice as large as the old one.... The optical peculiarities embodied in the new instrument will render it one of the three most powerful telescopes at present in existence.... The peculiar architectural feature of the building which is to shelter the new telescope is that its dome, of thirty-six feet diameter, will surmount a tower having a diameter of only thirty-one feet. Technically the form adopted is the surface generated by the revolution of an *involute of a circle.*"

after the rolling of $u\,x$ over an arc $u\,t$ we shall have $t\,x_1$ as the portion of the rolling line between x and the point of tangency, and $x\,y$ will have reached $x_1\,y_1$. If the rolling be continued y will evidently reach O. We see that $O\,y = u\,x$ and $O\,y_1 = t\,x_1$; but the lengths $u\,x$ and $t\,x_1$ are proportional to the angular movement of the rolling line about O, and as the spiral may be defined as that curve in which the length of a radius vector is directly proportional to the angle through which it has turned about the pole, the various positions of y are evidently points of such a curve.

189. Were the pole, O, given and a portion only of the spiral, we could draw a tangent at any point, y_1, by determining the circle on which the spiral could be trochoidally generated, then the instantaneous centre for the given position of the tracing-point, whence the normal and tangent would be derived in the usual way. The radius $O\,t$ of the base circle would equal $w\,y$,—the difference between two radii vectores $O\,y$ and $O\,z$ which include an angle of $57°.29+$, (the angle which at the centre of a circle subtends an arc equal to the radius). The instantaneous centre, t, would be the extremity of that radius which was perpendicular to $O\,y_1$. The normal would be $t\,y_1$ and the tangent $T\,T_1$ perpendicular to it.

190. This spiral is the proper outline for a cam to convert uniform rotary into uniform rectilinear motion, and when combined with an equal and opposite spiral gives the well known form called the *heart-cam*. As usually constructed the acting curve is not the true spiral but a curve whose points are at a constant distance from the theoretical outline equal to the radius of the friction-roller which is on the end of the piece to be raised. A small portion of such a "parallel curve" is indicated in the upper part of Fig. 107.

Fig. 108.



191. If a point travel on the surface of a cone so as to combine a uniform motion around the axis with a uniform motion toward the vertex it will trace a *conical helix* whose orthographic projection on the plane of the base will be a Spiral of Archimedes.

In Fig. 108 a top and front view are given of a cone and helix. The shaded portion is the *development* of the cone, that is, the area equal to the convex surface and which—if rolled up—would form the cone. To obtain the development draw an arc $A'\,G''\,A''$ of radius equal to an element. The convex surface of the cone will then be represented by the sector $A'\,O'\,A''$ whose angle θ may be found by the proportion $O\,A : O'\,A' :: \theta : 360°$, since the arc $A'\,G''\,A''$ must equal the entire circumference of the cone's base.

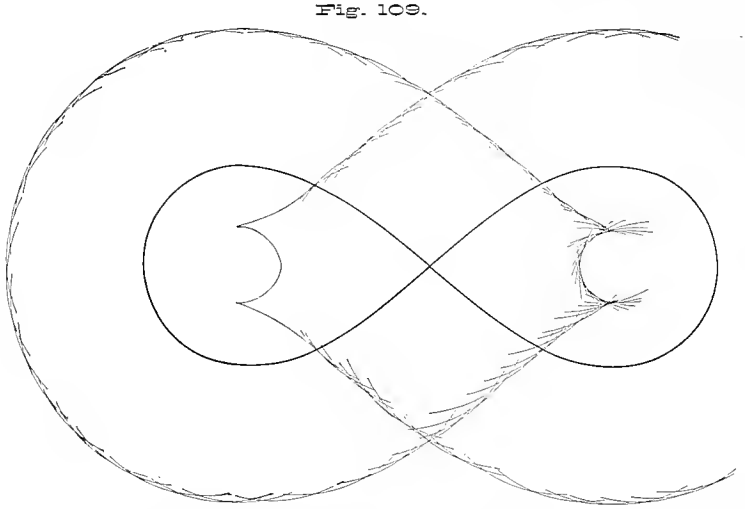
The student can make a paper model of the cone and helix by cutting out a sector of a circle, making allowance for an overlap on which to put the mucilage, as shown by the dotted lines $O'\,y$ and $y\,c\,z$ in the figure.

The *development* of the conical helix is evidently the same kind of spiral as its *orthographic projection.*

PARALLEL CURVES.

192. A *parallel curve* is one whose points are at a constant normal distance from some other curve. Parallel curves have not the same mathematical properties as those from which they are derived, except in the case of a circle; this can readily be seen from the cam figure under the last heading, in which a point, as S_1, of the true spiral, is located on a line from O which is by no means in the direction of the *normal* to the curve at S_1, upon which lies the point S_2 of the parallel curve.

Fig. 109.



Instead of actually determining the normals to a curve and on each laying off a constant distance we may draw many arcs of constant radius, having their centres on the original curve; the desired parallel will be tangent to all these arcs.

In strictly mathematical language the parallel curve is the *envelope* of a circle of constant radius whose centre is on the original curve. We may also define it as the locus of consecutive intersections of a system of equal circles having their centres on the original curve.

If on the *convex* side of the original the parallel will resemble it in form, but if *within* the two may be totally dissimilar. This is well illustrated by Fig. 109 in which the parallel to a Lemniscate is shown.

The student will obtain some interesting results by constructing the parallels to ellipses, parabolas and other plane curves.

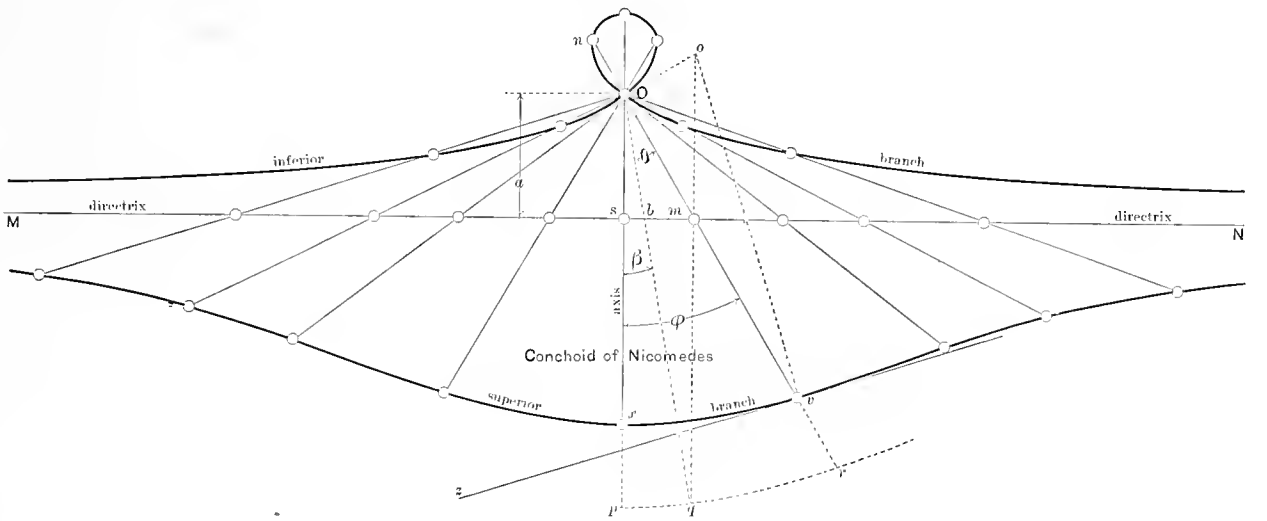
THE CONCHOID OF NICOMEDES.

193. The *Conchoid*, named after the Greek word for *shell*,* may be obtained by laying off a constant length on each side of a given line $M\,N$ (the *directrix*) upon radials through a fixed point or *pole*, O (Fig. 110). If $m\,v = m\,n = s\,x$ then v, n and x are points of the curve. Denote by a the distance of O from $M\,N$ and use c for the constant length to be laid off; then if $a < c$ there will be a loop in that branch of the curve which is nearest the pole,—the *inferior* branch. If $a = c$ the curve has a point or *cusp* at the pole. When $a > c$ the curve has an undulation or wave-form towards the pole.

* A series of curves much more closely resembling those of a shell can be obtained by tracing the paths of points on the piston-rod of an oscillating cylinder. See Arts. 157 and 158.

$O\,v = c + O\,m$; $O\,n = c - O\,m$; we may therefore express the relation to O of points on the curve by the equation $\rho = c \pm O\,m = c \pm a\ sec\ \phi$.

Fig. 110.



194. Mention has already been made of the fact that this was one of the curves invented for the purpose of solving the problem of the trisection of an angle. Were the angle $m\,O\,x$ (or ϕ) we would first draw $p\,q\,r$, the superior branch of a conchoid having the constant, c, equal to twice $O\,m$. A parallel from m to the axis will intersect the curve at q; the angle $p\,O\,q$ will then be one-third of ϕ: for since $b\,q = 2\,O\,m$ we have $m\,q = 2\,O\,m\ cos\ \beta$; also $m\,q : O\,m :: sin\ \theta : sin\ \beta$ (the sides of a triangle being proportional to the sines of the opposite angles): therefore $2\,O\,m\ cos\ \beta : O\,m :: sin\ \theta : sin\ \beta$, whence $sin\ \theta = 2\ sin\ \beta\ cos\ \beta = sin\ 2\ \beta$ (from known trigonometric relations). The angle ϕ is therefore equal to twice β, making the latter one-third of angle ϕ.

195. To draw a tangent and normal at any point r we find the instantaneous centre o on the principle that it is at the intersection of normals to the paths of two moving points of a line, the distance between said points remaining constant. The motion of r in tracing the curve is—at the instant considered—in the direction $O\,r$: $O\,o$ is therefore the normal. The point m of $O\,r$ is at the same moment moving along $M\,N$, for which $m\,o$ is the normal. Their intersection o is then the instantaneous centre and $o\,r$ the normal to the conchoid, with $r\,z$ perpendicular to $o\,r$ for the desired tangent.

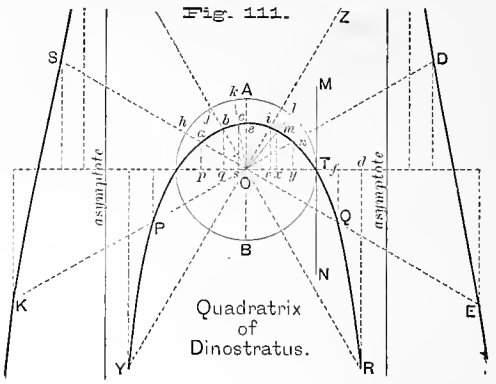
196. This interesting curve may be obtained as a plane section of one of the higher mathematical surfaces. If two non-intersecting lines—one vertical, the other horizontal—be taken as guiding lines or *directrices* of the motion of a third straight line whose inclination to a horizontal plane is to be constant, then every horizontal plane will cut *conchoids* from the surface thus generated, while every plane parallel to the directrices will cut *hyperbolas*. From the nature of its plane sections this surface is called the *Conchoidal Hyperboloid*.

THE QUADRATRIX OF DINOSTRATUS.

197. In Fig. 111 let the radius $O\,T$ rotate uniformly about the centre; simultaneously with its movement let $M\,N$ have a uniform motion parallel to itself, reaching $A\,B$ at the same time with radius $O\,T$: the locus of the intersection of $M\,N$ with the radius will be the *Quadratrix*. Points

exterior to the circle may be found by prolonging the radii while moving $M N$ away from $A B$. As the intersection of $M N$ with $O B$ is at infinity the former becomes an asymptote to the curve as often as it moves from the centre an additional amount equal to the diameter of the circle; the number of branches of the Quadratrix may therefore be infinite. It may be proved analytically that the curve crosses $O A$ at a distance from O equal to $2 r \div \pi$.

Fig. 111.



198. *To trisect an angle*, as $T O a$, by means of the Quadratrix draw the ordinate $a p$, trisect $p T$ by s and x and draw $s c$ and $x m$; radii $O c$ and $O m$ will then divide the angle as desired: for by the conditions of generation of the curve the line $M N$ takes three equidistant parallel positions while the radius describes three equal angles.

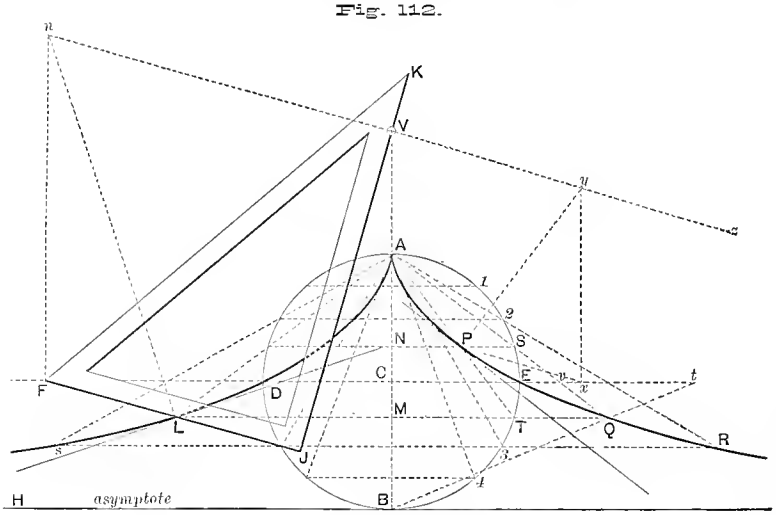
THE CISSOID OF DIOCLES.

199. This curve was devised for the purpose of obtaining two mean proportionals between two given quantities, by means of which one of the great problems of the Greek geometers—the duplication of the cube—might be effected.

The name was suggested by the Greek word for *ivy* since "the curve appears to mount along its asymptote in the same manner as that parasite plant climbs on the tall trunk of the pine."*

This was one of the first curves invented after the discovery of the conic sections. Let C (Fig. 112) be the centre of a circle, $A C E$ a right angle, $N S$ and $M T$ *any* pair of ordinates parallel to

Fig. 112.



and equidistant from $C E$; then either secant from A through the extremity of one ordinate will meet the other ordinate in a point of the cissoid; P and Q, for example, will be points of the curve.

The tangent to the circle at B will be an asymptote to the curve.

It is a somewhat interesting coincidence that the area between the cissoid and its asymptote is the same as that between a cycloid and its base, viz., three times that of the circle from which it is derived.

* Leslie. *Geometrical Analysis.* 1821.

200. Sir Isaac Newton devised the following method of obtaining a cissoid by continuous motion: Make $A\,V = A\,C$; then move a right-angled triangle, of base = $V\,C$, so that the vertex F travels along the line $D\,E$ while the edge $J\,K$ always passes through V; then the middle point, L, of the base $F\,J$, will trace a cissoid. This construction enables us readily to get the instantaneous centre and a tangent and normal; for $F\,n$ is normal to $F\,C$—the path of F, while $n\,z$ is normal to the motion of J toward $J\,V$; the instantaneous centre n is therefore at the intersection of these normals. For any other point as P we apply the same principle thus: With radius equal to $A\,C$ and from centre P obtain x; draw $P\,x$, then $V\,z$ parallel to it; a vertical from x will meet $V\,z$ at the instantaneous centre y, from which the normal and tangent result in the usual way.

201. Two quantities m and n will be mean proportionals between two other quantities a and b if $m^2 = n\,a$ and $n^2 = m\,b$; that is, if $m^3 = a^2\,b$ and if $n^3 = a\,b^2$.

If $b = 2\,a$ we will have m for the edge of a cube whose volume will be twice that of a^3, when $m^3 = a^2\,b$.

To get two mean proportionals between quantities r and b make the smaller, r, the radius of a circle from which derive a cissoid. Were $A\,P\,R$ the derived curve we would then make $C\,t$ equal to the second quantity, b, and draw $B\,t$, cutting the cissoid at Q. A line $A\,Q$ would cut off on $C\,t$ a distance $C\,v$ equal to m, one of the desired proportionals; for m^3 will then equal $r^2\,b$, as may be thus shown by means of similar triangles:—

$$C\,v : M\,Q :: C\,A : M\,A \text{ whence } C\,v^3 = \frac{r^3 . M\,Q^3}{M\,A^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$C\,t : M\,Q :: C\,B : B\,M \quad \text{“} \quad C\,t = \frac{r . M\,Q}{B\,M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$M\,Q : M\,A :: S\,N : A\,N :: \sqrt{A\,N . B\,N} : A\,N, \text{ whence } M\,Q = \frac{M\,A\sqrt{A\,N . B\,N}}{A\,N} \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{From (2) we have } M\,Q = \frac{B\,M\,(C\,t = b)}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{“ (3) “ “ } M\,Q^2 = \frac{M\,A^2\,(A\,N . B\,N)}{A\,N^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Replacing $M\,Q^3$ in equation (1) by the product of the second members of equations (4) and (5) gives $C\,v^3$ (i. e., m^3) $= r^2\,b$.

By interchanging r and b we obtain n, the other mean proportional; or it might be obtained by constructing similar triangles having a, b and m for sides.

THE TRACTRIX.

202. The *Tractrix* is the involute of the curve called the *Catenary* (later described) yet its usual construction is based on the fact that if a series of tangents be drawn to the curve the portions of such tangents between the points of tangency and a given line will be of the same length; or, in other words, the intercept on the tangent between the directrix and the curve will be constant. A practical and very close approximation to the theoretical curve is obtained by taking a radius $Q\,R$ (Fig. 113) and with a centre a, a short distance from R on $Q\,R$, obtaining b which is then joined with a. On $a\,b$ a centre, c is similarly taken, whence $c\,d$ is obtained. A sufficient repetition of this process will indicate the curve by its enveloping tangents; or a curve may actually be drawn tangent to all these lines. Could we take a, b, c, etc., as mathematically *consecutive points* the curve would be theoretically exact.

The line $Q\,S$ is an asymptote to the curve.

The area between the completed branch $R\,P\,S$ and the lines $Q\,R$ and $Q\,S$ would be equal to a quadrant of the circle on radius $Q\,R$.

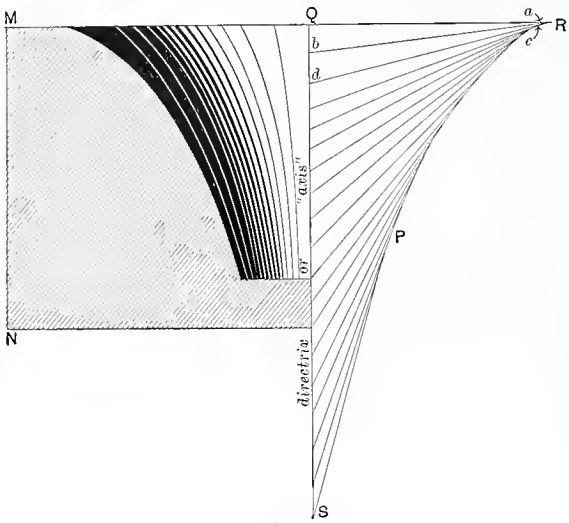


Fig. 113.

203. The surface generated by revolving the tractrix about its asymptote has been employed for the foot of a vertical spindle or shaft and is known as Schiele's Anti-Friction Pivot. The step for such a pivot is shown in sectional view in the left half of the figure. Theoretically the amount of work done in overcoming friction is the same on all equal areas of this surface.

In the case of a bearing of the usual kind, for a cylindrical spindle, although the *pressure* on each square inch of surface would be constant yet as unit areas at different distances from the centre would pass over very different amounts of space in one revolution, the wear upon them would be necessarily unequal. The *rationale* of the tractrix form will become evident from the following consideration:—If about to split a log, and having a choice of wedges, any boy would choose a thin one rather than one with a large angle, although he might not be able to prove by graphical statics the exact amount of advantage the one would have over the other. The theory is very simple, however, and the student may profitably be introduced to it. Suppose a ball, c, (Fig. 114) struck at the same instant by two others, a and b, moving at rates of six and eight feet a second respectively. On $a\,c$ and $b\,c$ prolonged take $c\,e$ and $c\,h$ equal, respectively, to *six* and *eight* units of some scale; complete the parallelogram having these lines as sides; then it is a well known principle in mechanics* that $c\,d$—the diagonal of this parallelogram—will not only represent the *direction* in which the ball c will move but also the *distance*—in feet, to the scale chosen—it will travel in one second. Obviously, then, to *balance* the effect of balls a and b upon c, a fourth would be necessary, moving from d toward c and traversing $d\,c$ in the same second that a and b travel, so that impact of all would occur simultaneously. These forces would be represented in direction and magnitude (to some scale) by the shaded triangle $c'\,d'\,c'$, which illustrates the very important theorem that if the three sides of a triangle—taken like $c'\,e'$, $e'\,d'$, $d'\,c'$, in such order as to bring one back to the initial vertex mentioned—represent in *magnitude* and *direction* three forces acting on one point, then these forces are balanced.

Fig. 114.

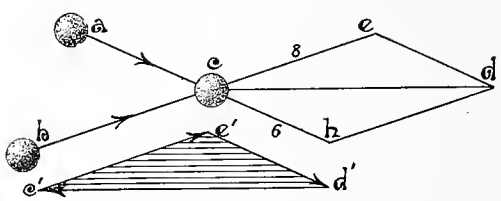
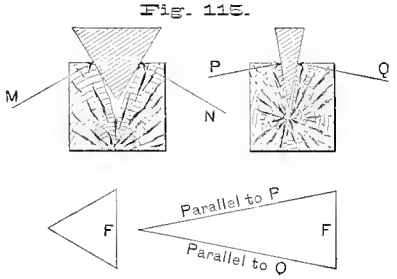


Fig. 115.



Constructing now a triangle of forces for a broad and thin wedge, (Fig. 115) and denoting the force of the supposed *equal blows* by F in each triangle, we see that the pressures are greater for the thin wedge than for the other; that is, the less the inclination to the vertical the greater the pressure. A pivot so shaped that as the *pressure* between it and its step *increased* the *area* to be traversed *diminished* would therefore, theoretically, be the ideal; and the rate of change of curvature of the tractrix as its generating point approaches the axis makes it, obviously, the correct form.

* For a demonstration the student may refer to Rankine's *Applied Mechanics*, Art. 51.

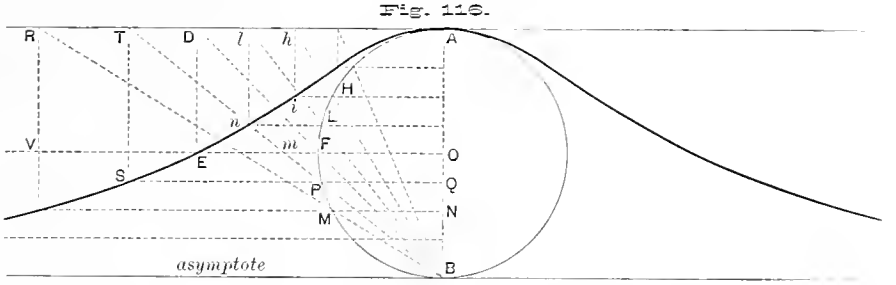
204. Navigator's charts are usually made by *Mercator's projection* (so-called, not being a *projection* in the ordinary sense, but with the extended signification alluded to in the remark in Art. 2). Maps thus constructed have this advantageous feature, that *rhumb lines* or *loxodromics*—the curves on a sphere that cut all meridians at the same angle—are represented as *straight lines*, which can only be the case if the meridians are indicated by *parallel* lines. The law of convergence of meridians on a sphere is, that the length of a degree of longitude at any latitude equals that of a degree on the equator multiplied by the cosine (see footnote, p. 31,) of the latitude; when the meridians are made *non-convergent* it is, therefore, manifestly necessary that the distance apart of originally equidistant parallels of latitude must increase at the same rate; or, otherwise stated, as on Mercator's chart degrees of longitude are all made equal, regardless of the latitude, the constant length representative of such degree bears a varying ratio to the actual arc on the sphere, being greater with the increase in latitude; but the greater the latitude the less its cosine or the greater its secant; hence lengths representative of degrees of latitude will increase with the secant of the latitude. Tables have been constructed giving the increments of the secant for each minute of latitude; but it is an interesting fact that they may be derived from the Tractrix thus:—Draw a circle with radius $Q\,R$, centre Q (Fig. 113); estimate latitude on such circle from R upward; the intercept on $Q\,S$ between consecutive tangents to the Tractrix will be the increment for the arc of latitude included between parallels to $Q\,S$, drawn through the points of contact of said pair of tangents.*

On the subject of map construction the student is referred to Craig's *Treatise on Projections*.

THE WITCH OF AGNISI.

205. If on any line $S\,Q$, perpendicular to the diameter of a circle, a point S be so located that $S\,Q : A\,B :: P\,Q : Q\,B$ then S will be a point of the curve called the *Witch of Agnisi*. Such point is evidently on the ordinate $P\,Q$ prolonged, and vertically below the intersection T of the tangent at A by the secant through P.

Fig. 116.



The point E, at the same level as the centre O, is a diameter's distance from the latter.

The tangent at B is an asymptote to the curve.

The area between the curve and its asymptote is four times that of the circle involved in its construction.

The *Witch*, also called the *Versiera*, was devised by Donna Maria Gaetana Agnisi, a brilliant Italian lady, who was appointed (1750) by Pope Benedict XIV. to the professorship of mathematics and philosophy in the University of Bologna.

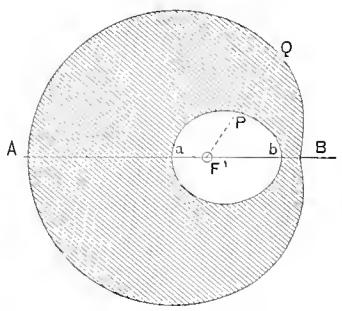
THE CARTESIAN OVAL.

206. This curve, also called simply a *Cartesian*, after its investigator, Descartes, has its points connected with two foci, F' and F'', by the relation $m\,\rho' \pm n\,\rho'' = k\,c$, in which c is the distance between the foci while m, n and k are constant factors.

* Leslie. *Geometrical Analysis*. Edinburgh, 1821.

Salmon states that we owe to Chasles the proof that a third focus may be found, sustaining the same relation, expressed by an equation of similar form. (See Art. 209.)

Fig. 117.

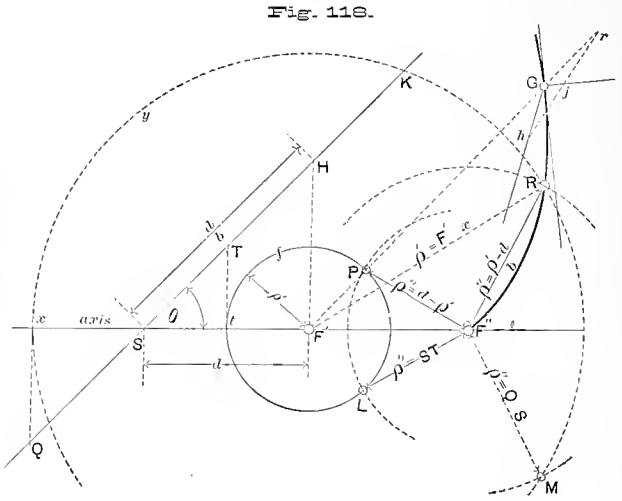


The Cartesian is symmetrical with respect to the *axis*—the line joining the foci.

207. To construct the curve from the first equation we may for convenience write $m\rho' \pm n\rho'' = kc$ in the form $\rho' \pm \frac{n}{m}\rho'' = \frac{kc}{m}$; or by denoting $\frac{n}{m}$ by b and $\frac{kc}{m}$ by d it takes the yet more simple form $\rho' \pm b\rho'' = d$. Then ρ'' will have two values according as the positive or negative sign is taken, being respectively $\frac{d-\rho'}{b}$ and $\frac{\rho'-d}{b}$; the former is for points on the *inner* of the *two* ovals that constitute a complete Cartesian, while the latter gives points on the *outer* curve.

To obtain $\rho'' = \frac{d-\rho'}{b}$ take F' and F''' (Fig. 118) as foci; $F'S = d$; SK at some random acute angle θ with the axis, and make $SH = \frac{d}{b}$; that is, make $F'S : SH :: b : 1$. Then from F' draw an arc tfP, of radius less than d, and cut it at P by an arc from centre F'', radius ST, Tt being a parallel to $F'H$; then P is a point of the inner oval; for $St = d - \rho'$ and $ST = \rho''$; therefore $\rho'' : d - \rho' :: \frac{d}{b} : d$ whence $\rho'' = \frac{d-\rho'}{b}$.

Fig. 118.



208. If an arc xyk be drawn from F'' with radius, $F''x$, *greater than* d, we may find the second value of ρ'', viz., $\frac{\rho'-d}{b}$ by drawing xQ parallel to $F'H$ to meet HS prolonged; for QS will equal $\frac{\rho'-d}{b}$, in which $\rho' = F'x$. Again using F'' as a centre, and a radius $QS = \rho''$, gives points R and M of the larger oval.

The following are the values for the focal radii to the four points where the ovals cut the axes. (See Fig. 117.)

For A, $\rho'' = \frac{\rho'-d}{b} = c + \rho'$ whence $\rho' = F'A = \frac{d + bc}{1-b}$

" a, $\rho'' = \frac{d-\rho'}{b} = c + \rho'$ " $\rho' = F'a = \frac{d - bc}{1+b}$

" B, $\rho'' = \frac{\rho'-d}{b} = c - \rho'$ " $\rho' = F'B = \frac{d + bc}{1+b}$

" b, $\rho'' = \frac{d-\rho'}{b} = c - \rho'$ " $\rho' = F'b = \frac{d - bc}{1-b}$

The construction-arcs for the outer oval must evidently have radii *between* the values of ρ' for A and B above; and for the inner oval *between* those of a and b.

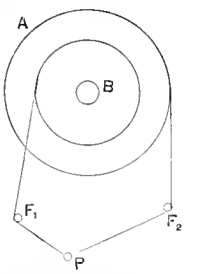
The numerical values from which Fig. 118 was constructed were $m = 3$; $n = 2$; $c = 1$; $k = 3$.

209. *The Third Focus.* Figure 118 illustrates a special case, but in general the method of finding a third focus, F''' (not shown), would be to draw a random secant $F' r$ through F' and note the points P and G in which it cuts the ovals—these to be taken on the same side of F', as two other points of intersection are possible; a circle through P, G and F'' would cut the axis in the new focus sought. Then denoting by C the distance $F' F'''$, we would find the factors of the original equation appearing in a new order, thus, $k\rho' \pm n\rho''' = mC$, which—for purposes of construction—may be written $\rho' \pm b'\rho''' = d'$.

If obtained from the foci F''' and F'''' the relation would be $m\rho''' - k\rho'' = \pm nC'$, in which C' equals $F''F'''$. Writing this in the form $\rho''' - B\rho'' = \pm D$ we have the following interesting cases: (a) an *ellipse* for D positive and $B = -1$; (b) an *hyperbola* for D positive and $B = +1$; (c) a *limaçon* for $D = C'B$; (d) a *cardioid* for $B = +1$ and $D = C'$.

210. The following method of drawing a Cartesian by continuous motion was devised by Prof. Hammond:—A string is wound, as shown, around two pulleys turning on a common axis; a pencil at P holds the string taut around smooth pegs placed at random at F_1 and F_2; if the wheels be turned with the same angular velocity and the pencil does not slip on the string it will trace a Cartesian having F_1 and F_2 as foci.[1]

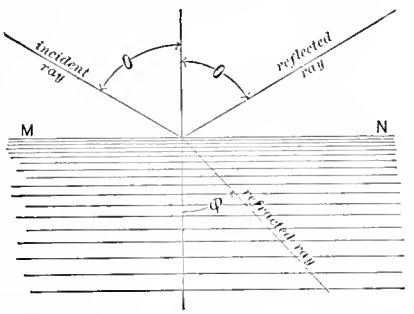
Fig. 119.



If the pulleys are *equal* the Cartesian will become an *ellipse*; if both threads are wound *the same way* around *either one* of the wheels the resulting curve will be an *hyperbola*.

211. It is a well-known fact that the incident and reflected ray make equal angles with the normal to a reflecting surface. If the latter is *curved* then each reflected ray cuts the one next to it, their consecutive intersections giving a curve called a *caustic by reflection*. Probably all have occasionally noticed such a curve on the surface of the milk in a glass, when the light was properly placed. If the reflecting curve is a *circle* the caustic is the *evolute of a limaçon*.

Fig. 120.



In passing from one medium *into* another, as from air into water, the deflection which a ray of light undergoes is called *refraction*, and for the same media the ratio of the *sines* of the angles of incidence and refraction (θ and ϕ, Fig. 120,) is constant. The consecutive intersections of *refracted* rays give also a *caustic*, which, for a *circle*, is the *evolute* of a *Cartesian Oval*. The proof of this statement[2] involves the property upon which is based the most convenient method of drawing a tangent to the Cartesian, viz., that the normal at any point divides the angle between the focal radii into parts whose sines are proportional to the factors of those radii in the equation. If, then, we have obtained a point G on the outer oval from the relation $m\rho' \pm n\rho'' = kc$ we may obtain the tangent at G by laying off on ρ' and ρ'' distances proportional to m and n, as Gr and Gh, Fig. 118, then bisecting rh at j and drawing the normal Gj, to which the desired tangent is a perpendicular.

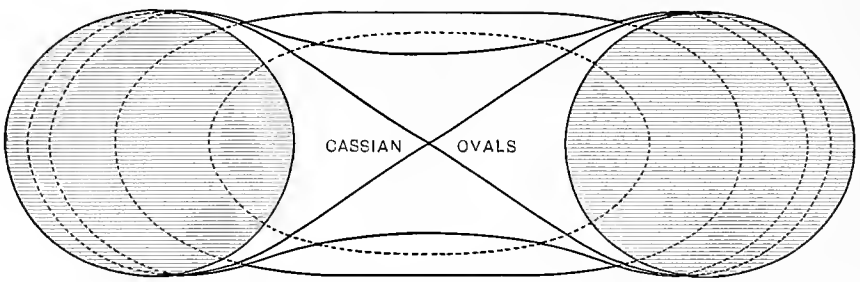
At a point on the inner oval the distance would not be laid off on a focal radius *produced*, as in the case illustrated.

[1] *American Journal of Mathematics*, 1878. [2] Salmon. *Higher Plane Curves.* Art. 117.

CASSIAN OVALS.

212. In the *Cassian Ovals* or *Ovals of Cassini* the points are connected with two foci by the relation $\rho' \rho'' = k^2$, i. e., the *product* of the focal radii is equal to some perfect square. These curves have already been alluded to in Art. 114 as plane sections of the annular torus, taken parallel to its axis.

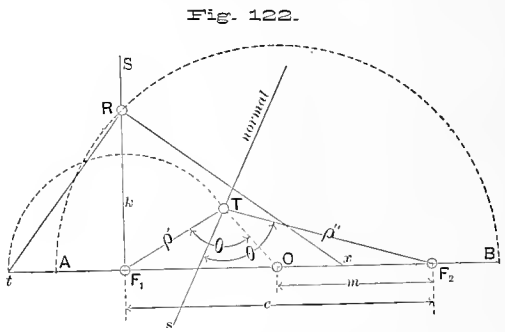
Fig. 121.



In Art. 158 one form — the Lemniscate — receives special treatment. For it the constant k^2 must equal m^2, the square of half the distance between the foci. When k is less than m the curve becomes two separate ovals.

213. The general construction depends on the fact that in any semicircle the square of an ordinate equals the product of the segments into which it divides the diameter. In Fig. 122 take F_1 and F_2 as the foci, erect a perpendicular $F_1 S$ to the axis $F_1 F_2$ and on it lay off $F_1 R$ equal to the constant k. Bisect $F_1 F_2$ at O and draw a semicircle of radius $O R$. This cuts the axis at A and B, the extreme points of the curve; for $k^2 = F_1 A \times F_1 B$. Any other point T may be obtained by drawing from F_1 a circular arc of radius $F_1 t$ greater than $F_1 A$; draw $t R$, then $R x$ perpendicular to it; $x F_1$ will then be the ρ'' and $F_1 t$ the ρ' for four points of the curve, which will be at the intersection of arcs struck from F_1 and F_2 as centres and with those radii.

Fig. 122.



To get a normal at any point T draw $O T$, then make angle $F_2 T s = \theta = F_1 T O$; $T s$ will be the desired line.

THE CATENARY.

214. If a flexible chain, cable or string, of uniform weight per unit of length, be freely suspended by its extremities, the curve which it takes under the action of gravity is called a *Catenary*, from *catena*, *a chain*.

A simple and practical method of obtaining a catenary on the drawing-board would be to insert two pins in the board, in the desired relative position of the points of suspension, and then attach to them a string of the desired length. By holding the board vertically the string would assume the catenary, whose points could then be located with the pencil and joined in the usual manner with the irregular curve. Otherwise, if its points are to be located by means of an equation, we take axes in the plane of the curve, the y-axis (Fig. 123) being a vertical line through the lowest point T of the catenary, while the x-axis is a horizontal line at the distance m below T. The quantity m is called the *parameter* of the curve and is equal to the length of string which represents the tension at the lowest point.

The equation of the catenary[1] is then $y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$ in which e is the *base* of Napierian logarithms[2] and has the numerical value 2.7182818 +.

By taking successive values of x equal to m, $2\,m$, $3\,m$, etc., we get the following values for y:—

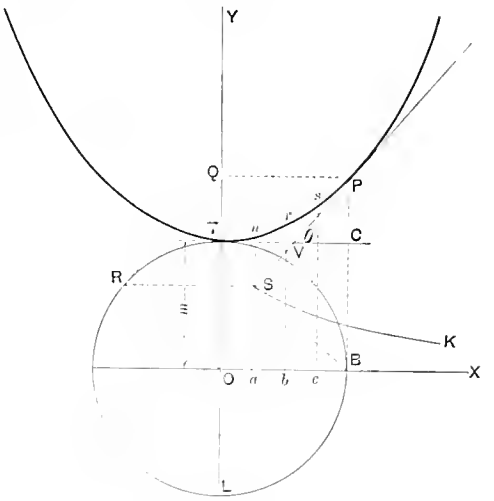
$x = \quad m \ldots y = \frac{m}{2}\left(e + \frac{1}{e}\right)$ which for m = *unity* becomes 1.54308

$x = 2\,m \ldots y = \frac{m}{2}\left(e^2 + \frac{1}{e^2}\right)$ " " " " " 3.76217

$x = 3\,m \ldots y = \frac{m}{2}\left(e^3 + \frac{1}{e^3}\right)$ " " " " " 10.0676

$x = 4\,m \ldots y = \frac{m}{2}\left(e^4 + \frac{1}{e^4}\right)$ " " " " " 27.308

Fig. 123.



To construct the curve we therefore draw an arc of radius $O\,B = m$, giving T on the axis of y as the lowest point of the curve.

For $x = O\,B = m$ we have $y = B\,P = 1.54308$; for $x = O\,a = \frac{m}{4}$ we have $y = a\,n = 1.03142$.

The tension at any point P is equal to the weight of a piece of rope of length $B\,P = P\,C + m$.

At the lowest point the tangent is horizontal. The length of any arc $T\,P$ is proportional to the angle θ between $T\,C$ and the tangent $P\,V$ at the upper extremity of the arc.

215. If a circle $R\,L\,B$ be drawn, of radius equal to m, it may be shown analytically that tangents $P\,S$ and $Q\,R$, to catenary and circle respectively, from points at the same level, will be parallel: also that $P\,S$ equals the catenary-arc $P\,r\,T$: S therefore traces the involute of the catenary, and as $S\,B$ always equals $R\,O$ and remains perpendicular to $P\,S$ (angle $O\,R\,Q$ being always 90°) we have the curve $T\,S\,K$ fulfilling the conditions of a *tractrix*. (See Art. 202.)

If a parabola, having a focal distance m, roll on a straight line, the focus will trace a catenary having m for its parameter.

The catenary was mistaken by Galileo for a parabola. In 1669 Jungius proved it to be neither a parabola nor hyperbola, but it was not till 1691 that its exact mathematical nature was known, being then established by James Bernouilli.

THE LOGARITHMIC OR EQUIANGULAR SPIRAL.

216. In Fig. 124 we have the curve called the *Logarithmic Spiral*. Its usual construction is based on the property that any radius vector, as ρ, which bisects the angle between two other radii, $O\,M$ and $O\,N$, is a mean proportional between them; i. e., $\rho^2 = O\,S^2 = O\,M \times O\,N$.

If M and G are points of the spiral we may find an intermediate point K by drawing the ordinate $O\,K$ to a semicircle of diameter $O\,M + O\,G$. A perpendicular through G to $G\,K$ will then give D, another point of the curve, and this construction may be repeated indefinitely.

Radii making equal angles with each other are evidently in geometrical progression.

The curve never reaches the pole.

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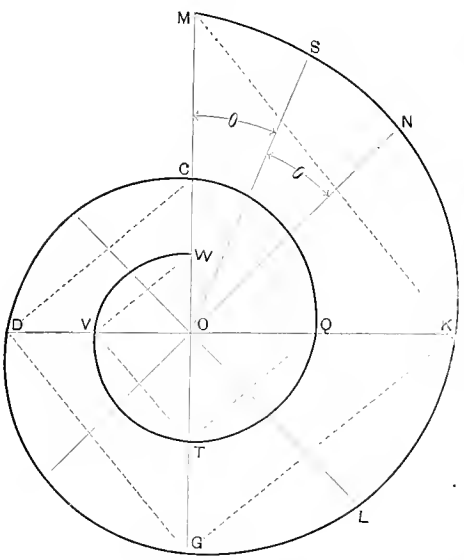
[1] Rankine. *Applied Mechanics*, Art. 175.

[2] In the expression $10^2 = 100$ the quantity "2" is called the *logarithm* of 100, it being the exponent of the power to which 10 must be raised to give 100. Similarly 2 would be the logarithm of 64 were 8 the *base* or number to be raised to the power indicated.

This spiral is often called *Equiangular* from the fact that the angle is always the same between a radius vector and the tangent at its extremity. Upon this property is based its use as the outline for spiral cams and for lobed wheels.

Fig. 124.



The name *logarithmic spiral* is based on the property that the angle of revolution is proportional to the logarithm of the radius vector. This is expressed by $\rho = a^{\theta}$, in which θ is the varying angle and a is some arbitrary constant.

To construct a tangent by calculation divide the hyperbolic logarithm[1] of the ratio $O\,M : O\,K$ (which are any two radii whose values are known) by the angle between these radii, expressed in circular measure[2]; the quotient will be the tangent of the constant angle of obliquity of the spiral.

217. Among the more interesting properties of this curve are the following:—

Its involute is an equal logarithmic spiral.

Were a light placed at the pole, the caustic—whether by reflection or refraction—would be a logarithmic spiral.

The discovery of these properties of recurrence led James Bernouilli to direct that this spiral be engraved on his tomb, with the inscription—*Eadem Mutata Resurgo*, which, freely translated, is—*I shall arise the same, though changed.*

Kepler discovered that the orbits of the planets and comets were conic sections having a focus at the centre of the sun. Newton proved that they would have described logarithmic spirals as they travelled out into space had the attraction of gravitation been inversely as the *cube* instead of the *square* of the distance.

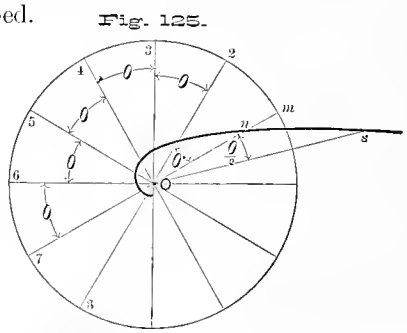
THE HYPERBOLIC OR RECIPROCAL SPIRAL.

218. In this spiral the length of a radius vector is in inverse ratio to the angle through which it turns.

Like the logarithmic spiral it has an infinite number of convolutions about the pole, which it never reaches.

The invention of this curve is attributed to James Bernouilli, who showed that Newton's conclusions as to the logarithmic spiral (see Art. 217) would also hold for the hyperbolic spiral, the initial velocity of projection determining which trajectory was described.

Fig. 125.



To obtain points of the curve divide a circle $m\,5\,8$ (Fig. 125) into any number of equal parts, and on some initial radius $O\,m$ lay off some unit, as an inch; on the second radius $O\,2$ take $\frac{O\,n}{2}$; on the third $\frac{O\,n}{3}$, etc. For one-half the angle θ the radius vector would evidently be $2\,O\,n$, giving a point s outside the circle.

The equation to the curve is $\frac{1}{r} = a\,\theta$, in which r is the

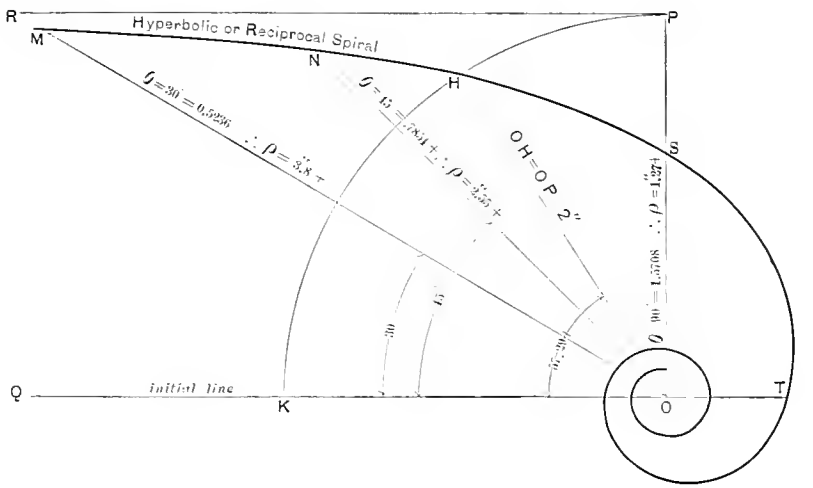
[1] To get the hyperbolic logarithm of a number multiply its common logarithm by 2.3026.

[2] In circular measure $360° = 2\,\pi\,r$ which for $r = 1$ becomes 6.28318; $180° = 3.14159$; $90° = 1.5708$; $60° = 1.0472$; $45° = 0.7854$; $30° = 0.5236$; $1° = 0.0174533$.

radius vector, a some numerical constant, and θ is the angular rotation of r (in circular measure) estimated from some initial line.

The curve has an asymptote parallel to the initial line and at a distance from it equal to $\frac{1}{a}$ units.

Fig. 126.



To construct the spiral from its equation take O as the pole (Fig. 126); $O\,Q$ as the initial line: a for convenience, some fraction, as $\frac{1}{4}$: and as our *unit* some quantity, say half an inch, that will make $\frac{1}{a}$ of convenient size. Then taking $Q\,O$ as the initial line make $O\,P = \frac{1}{a} = 2''$ and draw $P\,R$ parallel to $O\,Q$ for the asymptote. For $\theta = 1$, that is, for *arc* $K\,H$ = *radius* $O\,H$ we have $r = \frac{1}{a} = 2''$, giving H,— one point of the spiral. Writing the equation in the form $r = \frac{1}{a} \cdot \frac{1}{\theta}$ and expressing various values of θ in circular measure we get the following:—

$\theta = 30° = 0.5236$: $r = O\,M = 3''.8 +$: $\theta = 45° = 0.7854$; $r = O\,N = 2''.55$:

$\theta = 90° = 1.5708$; $r = O\,S = 1''.2 +$: $\theta = 180° = 3.14159$; $r = O\,T = .6366$, etc.

The tangent to the curve at any point makes with the radius vector an angle ϕ which is found by analysis to sustain to the angle θ the following trigonometrical relation, $\tan \phi = \theta$; the circular measure of θ may therefore be found in a table of natural tangents and the corresponding value of ϕ obtained.

THE LITUUS.

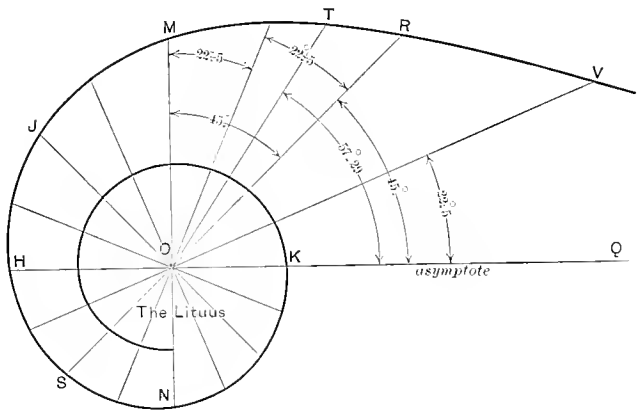
219. The spiral in which the radius vector is inversely proportional to the square root of the angle through which it has revolved is called the *Lituus*. This relation is shown by the equation $r = \frac{1}{a\sqrt{\theta}}$, also written $a^2\,\theta = \frac{1}{r^2}$.

When $\theta = 0$ we find $r = \infty$, making the initial line an asymptote to the curve.

In Fig. 127 take $O\,Q$ as the initial line, O as the pole, $a = 2$, and our unit a three-inch line: then $\frac{1}{a} = 1\frac{1}{2}''$.

For $\theta = 90° = \pi$ (in circular measure 1.5708) we have $r = O\,M = 1''.2+$. For $\theta = 1$ we have the radius $O\,T$ making an angle of $57°.29+$ with the initial line, and in length equal to $\frac{1}{a}$ units, i. e., $1\frac{1}{2}''$. For $\theta = 45° = \frac{\pi}{4}$ (or 0.7854) r will be $O\,R = 1''.7+$. Then $O\,H = \frac{O\,R}{2}$; for in rotating to $O\,H$ the radius vector passes over four 45° angles, and the radius must therefore be one-half what it was for the first 45° described. Similarly $O\,K = \frac{O\,M}{2}$; $O\,M = \frac{O\,V}{2}$, etc.; this relation enabling the student to locate any number of points.

Fig 127.



To draw a tangent to the curve we employ the relation $\tan \phi = 2\,\theta$, ϕ being the angle made by the tangent line with the radius vector, while θ is the angular rotation of the latter, in circular measure.

CHAPTER VI.

TINTING — FLAT AND GRADUATED. — MASONRY, TILING, WOOD GRAINING, RIVER-BEDS AND OTHER SECTIONS, WITH BRUSH ALONE OR IN COMBINED BRUSH AND LINE WORK.

220. Brush-work, with ink or colors, is either *flat* or *graduated;* the former gives the effect of a flat surface parallel to the paper on which the drawing is made, while graded tints either show curvature or — if indicating flat surfaces — represent them as inclined to the paper, i. e., to the plane of projection: for either, the paper should be, as previously stated (Arts. 41 and 44) *cold-pressed* and *stretched.*

The surface to be tinted should not be abraded by sponge, knife or rubber.

221. The liquid employed for tinting must be free from sediment; or, at least, the latter, if present, must be allowed to settle and the brush dipped only in the clear portion at the top. Tints may, therefore, best be mixed in an artist's water-glass rather than in anything shallower. In case of several colors mixed together, however, it would be necessary to thoroughly stir up the tint each time before taking a brushful.

A tint prepared from a *cake* of high-grade India ink is far superior to any that can be made by using the ready-made liquid drawing inks.

222. The size of brush should bear some relation to that of the surface to be tinted; large brushes for large surfaces and vice versa. The customary error of beginners is to use too small and too dry a brush for tinting, and the reverse for shading.

223. Harsh outlines are to be avoided in brush work, especially in handsomely shaded drawings in which, if sharply defined, they would detract from the general effect. This will become evident on comparing the spheres in Figs. 1 and 4 of Plate II.

If tinting and shading can be successfully done with only *pencilled* limits there is then no excuse for inking the boundaries; but if, for the sake of definiteness, the outlines are inked at all it should be *before* the tinting and in the finest of lines, preferably of "water-proof" ink; although any ink will do provided a soft sponge and plenty of clean water be applied to remove any excess that will "run." The sponge is also to be the main reliance of the draughtsman for the correction of errors in brush work; the water, however, and not the friction to be the active agent. An entire tint may be removed in this way in case it seems desirable.

224. When beginning work incline the board at a small angle so that the tint will flow down after the brush. For a *flat*, that is, a *uniform* tint, start at the upper outline of the surface to be covered and with the brush full, yet not surcharged—which would prevent its coming to a good point—pass lightly along from left to right and on the return carry the tint down a little farther, making short, quick strokes, with the brush held almost perpendicular to the paper. Advance the tint as evenly as possible along a horizontal line: work quickly *between* outlines, but more slowly *along* outlines, as one should never overrun the latter and then resort to "trimming" to conceal lack of skill. It is possible for any one with care and practice to tint *to* yet not *over* boundaries.

The advancing edge of the tint must not dry until the lower boundary is reached.

No portion of the paper, however small, should be missed as the tint advances, as the work is likely to be spoiled by retouching.

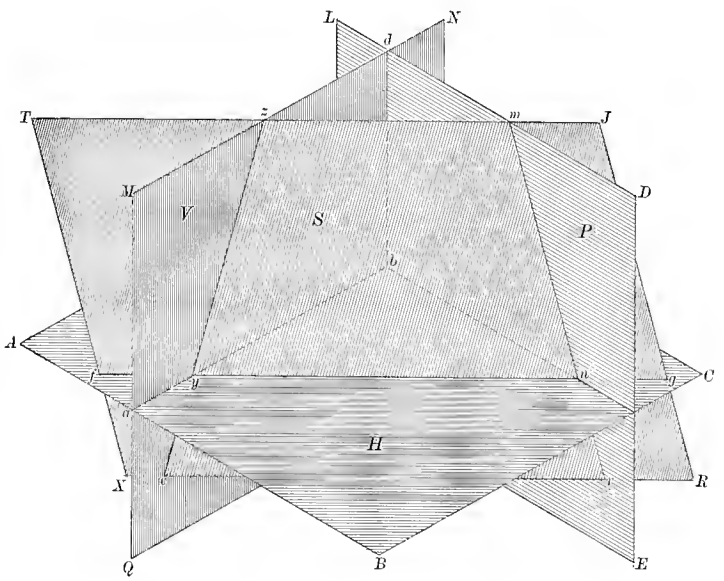
Should any excess of tint be found along the lower edge of the figure it should be absorbed by the brush, after first removing the latter's surplus by means of blotting paper.

To get a dark effect several medium tints laid on in succession, each one drying before the next is applied, give better results than one dark one.

The heightened effect described in Art. 72, viz., a line of light on the upper and left-hand edges, may be obtained either (a) by ruling a broad line of *tint* with the drawing-pen at the desired distance from the outline and instantly, before it dries, tinting from it with the brush; or (b) by ruling the line with the pen and thick Chinese White.

225. A tint will spread much more evenly on a large surface if the paper be first slightly dampened with clean water. As the tint will follow the water, the latter should be limited exactly to the intended outlines of the final tint.

Fig. 128.



226. Of the colors frequently used by engineers and architects those which work best for flat effects are carmine, Prussian blue, burnt sienna and Payne's gray. Sepia and Gamboge are, fortunately, rarely required for even tints; but the former works ideally for *shading* by the "dry" process described in the next article; and its rich brown gives effects unapproachable with anything else. It has, however, this peculiarity, that repeated touches upon a spot to make it darker produce the opposite effect unless enough time elapses between strokes to allow each addition to dry thoroughly.

227. For elementary practice with the brush the student should lay flat washes, in India tints, on from six to ten rectangles, of sizes between $2'' \times 6''$ and $6'' \times 10''$. If successful with these his next work may be the reproduction of Fig. 128, in which H, V, P and S denote horizontal, vertical, profile and section planes respectively. The figure should be considerably enlarged.

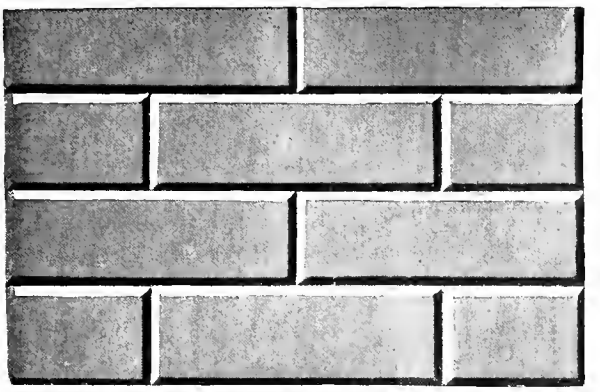
The plane V may have two washes of India ink; H one of Prussian blue; P one of burnt sienna and S one of carmine.

The edges of H, V and P are either vertical or inclined 30° to the horizontal.

For the section-plane assume n and m at pleasure, giving direction $n\,m$, to which $J\,R$ and $T\,X$ are parallel. A horizontal, $m\,z$, through m gives z. From n a horizontal, $n\,y$, gives y on $a\,b$. Joining y with z gives the "trace" of S on V.

228. Figures 129 and 130 illustrate the use of the brush in the representation of masonry. The former may be altogether in ink tints, or in medium burnt umber for the front rectangle of

Fig. 129.

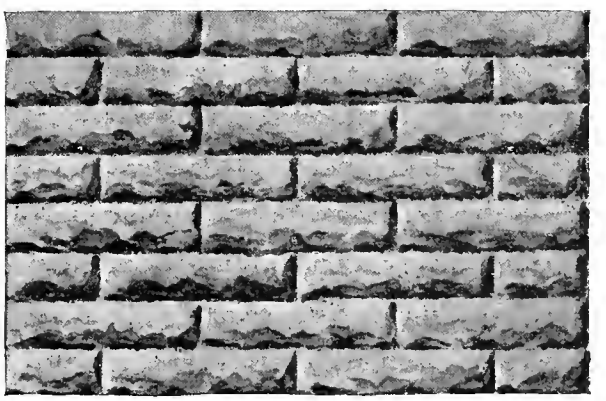


each stone, and dark tint of the same, directly from the cake, for the bevel. Lightly pencilled limits of bevel and rectangle will be needed; no inked outlines required or desirable.

The last remark applies also to Fig. 130, in which "quarry-faced" ashlar masonry is represented. If properly done, in either burnt umber or sepia, this gives a result of great beauty, especially effective on the piers of a large bridge drawing.

The darker portions are tinted directly from the cake, and are purposely made irregular and "jagged" to reproduce as closely as possible the fractured appearance of the stone.

Fig. 130.

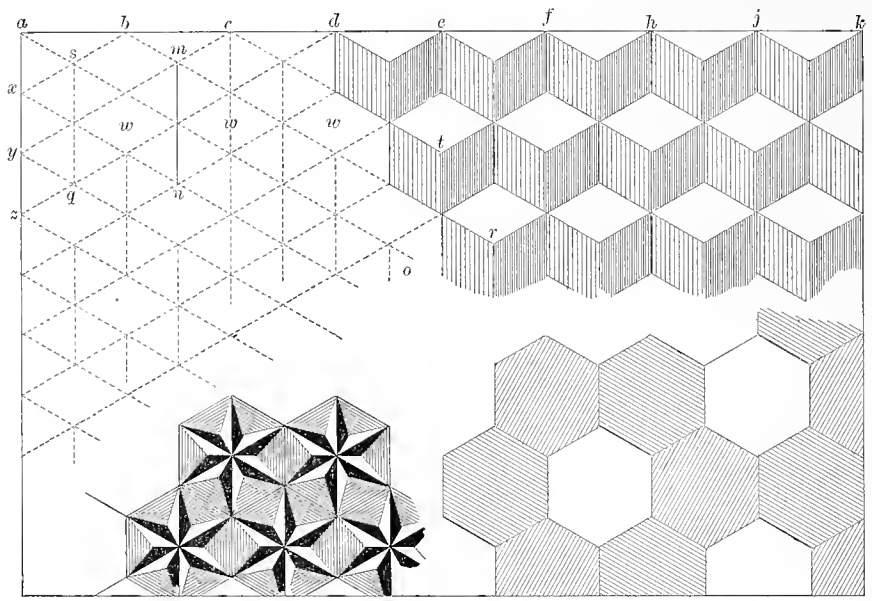


When an "over-hang" or jutting portion is to be represented two brushes are required, one with a medium tint, the other with the thick color, as before; an irregular line being made with the latter the tint is softened out *on the lower side* with the point of the brush having the lighter tint. A light wash of the intended tone of the whole mass is quickly laid over each stone, either *before* or *after* the irregularities are represented, according as an exceedingly angular or a somewhat softened and rounded effect is desired.

229. Designs in tiling are excellent exercises not only for brush work in flat tints but also—in their preliminary construction—in precision of line work. The superbly illustrated catalogues of the Minton Tile Works are, unfortunately, not accessible by all students, illustrating as they do, the finest and most varied work in this line, both of designer and chromo-lithographer; but it is quite within the bounds of possibility for the careful draughtsman to closely approach if not equal the standard and general appearance of their work, and as suggestions therefor Figs. 131 and 132 are presented.

230. In Fig. 131 the upper boundary, *a d h k*, of a rectangle is divided at *a*, *b*, *c*, etc., into equal spaces, and through each point of division two lines are drawn with the 30° triangle, as *b x* and *b r* through *b*. The oblique lines terminate on the sides and lower line of the rectangle. If the work is accurate—and it is worthless if not—any vertical line as *m n*, drawn through the intersection, *m*, of a pair of oblique lines, will pass through the intersection of a series of such pairs.

Fig. 131.



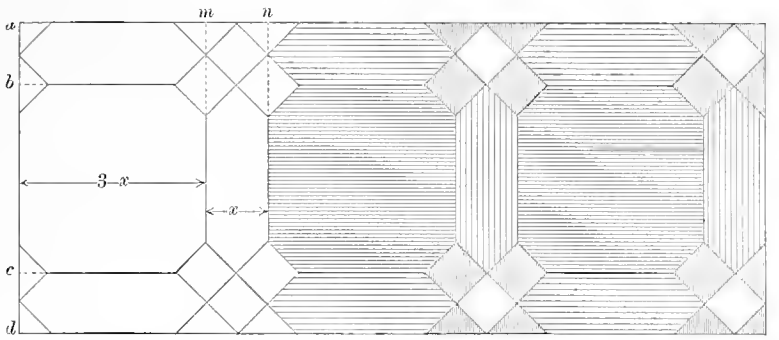
The figure shows three of the possible designs whose construction is based on the dotted lines of the figure. For that at the top and right, in which *horizontal* rows of rhombi are left white, we draw vertical lines as *s q* and *m n* from the lower vertex of each intended white rhombus, continuing it over two rhombi, when another white one will be reached. The dark faces of the design are to be finally in solid black, previous to which the lighter faces should be tinted with some drab or brown tint. The pencilled construction lines would necessarily be erased before the tint was laid on.

The most opaque effect in colors is obtained by mixing a large proportion of Chinese white with the water color, making what is called by artists a "body color." Such a mixture gives a result in marked contrast with the transparent effect of the usual wash; but the amount of white used should be sufficient to make the tint in reality a *paste*, and no more should be taken on the brush at one time than is needed to cover one figure.

Sepia and Chinese white, mixed in the proper proportions, give a tint which contrasts most agreeably with the black and white of the remainder of the figure. The star design and the hexagons in the lower right-hand corner result from extensions or modifications of the construction just described, which will become evident on careful inspection.

231. Fig. 132 is a Minton design with which many are familiar and which affords opportunity for considerable variety in finish. Its construction is almost self-evident. The equal spaces *a b*, *c d*, *m n*—which may be any width, *x*,—alternate with other equal spaces *b c* which may preferably be about 3 *x* in width. Lines at 45°, as indicated, complete the preliminaries to tinting.

Fig. 132.



The octagons may be in Prussian blue, the hexagons in carmine and the remainder in white and black, as shown; or browns and drabs may be employed for more subdued effects.

SHADING.

232. For *shading*, by graduated tints, provide a glass of clear water in addition to the tint; also an ample supply of blotting paper.

The water-color or ink tint may be considerably darker than for flat tinting; in fact, the darker it is, provided it is clear, the more rapidly can the desired effect be obtained.

The brush must contain much less liquid than for flat work.

Lay a narrow band of tint quickly along the part that is to be the darkest, then dip the brush into clear water and immediately apply it to the blotter, both to bring it to a good point and to remove the surplus tint. With the now once-diluted tint carry the advancing edge of the band slightly farther. Repeat the operation until the tint is no longer discernible as such.

The process may be repeated from the same starting point as many times as necessary to produce the desired effect; but the work should be allowed to dry each time before laying on a new tint.

Any irregularities or streaks can easily be removed, after the work dries, by retouching or "stippling" with the point of a fine brush that contains *but little tint*—scarcely more than enough to enable the brush to retain its point. For small work, as the shading of rivets, rods, etc., the process just mentioned, which is also called "dry shading," is especially adapted, and although somewhat tedious gives the handsomest effects possible to the draughtsman.

233. Where a good, *general* effect is wanted, to be obtained in less time than would be required for the preceding processes, the method by over-lapping flat tints may be adopted. A narrow band of dark tint is first laid over the part to be the darkest. When dry this is overlaid by a broader band of lighter tint. A yet lighter wash follows, beginning on the dark portion and extending still farther than its predecessor. The process is repeated with further diluted tints until the desired effect is obtained.

Faintly-pencilled lines may be drawn at the outset as limits for the edges of the tints.

This method is better adapted for large work, that is not to be closely scrutinized, than for drawings that deserve a high degree of finish.

234. As to the relative position and gradation of the lights and shades on a figure the student is referred to Arts. 78 and 79 and the chapter on shadows; also to the figures of Plate II, which may serve as examples to be imitated while the learner is acquiring facility in the use of the brush and before entering upon constructive work in shades and shadows. Fig. 3 of Plate II may be undertaken first, and the contrast made yet greater between the upper and lower boundaries. Fig. 1 (Plate II) requires no explanation. In Fig. 133, we have a wood-cut of a sphere with the theoretical dark or "shade" line more sharply defined than in the spheres on the plate.

Fig. 133.

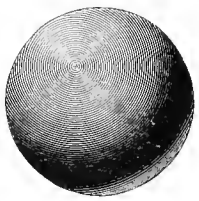
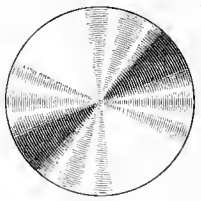


Fig. 134.



A drawing of the end of a highly-polished revolving shaft or even of an ordinary metallic disc would be shaded as in Fig. 134.

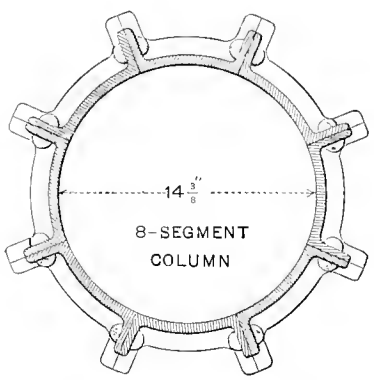
Fig. 2 (Plate II) represents the triangular-threaded screw, its oblique surfaces being, in mathematical language, *warped helicoids*, generated by a moving straight line, one end of which travels *along the axis* of a cylinder while the other end traces or follows a helix *on* the cylinder.

The construction of the helix having already been given (Art. 120) the outlines can readily be drawn. The method of exactly locating the shadow and shade lines will be found in the chapter on shadows.

Fig. 4 (Plate II) when compared with Fig. 91 illustrates the possibilities as to the representation of interesting mathematical relations. The fact may again be mentioned, on the principle of "line upon line," as also for the benefit of any who may not have read all that has preceded, that the spheres in the cone are tangent to the oblique plane at the *foci* of the elliptical section. The peculiar dotted effect in this figure is due to the fact that the original drawing, of which this is a photographic reproduction by the gelatine process, was made with a lithographic crayon upon a special pebbled paper much used by lithographers. The original of Fig. 1, on the other hand, was a brush-shaded sphere on Whatman's paper.

235. Fig. 5 (Plate II) shows a "Phœnix column," the strongest form of iron for a given weight, for sustaining compression. The student is familiar with it as an element of outdoor construction in bridges, elevated railroads, etc.; and it is being increasingly employed in indoor work, notable examples as to size being those of the New York *World* building.

Fig. 135.

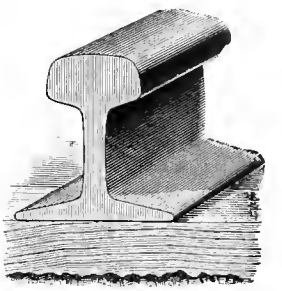


By drawing first an end view of a Phœnix column, similar to that of Fig. 135, we can readily derive an oblique view like that of the plate, by including it between parallels from all points of the former. The proportions of the columns are obtainable from the tables of the company.

Fig. 135 is a cross-section of the 8-segment column, the shaded portion showing the minimum and the other lines the maximum size for the same inside diameter.

In a later chapter the proportions of other forms of structural iron will be found. Short lengths of any of these, if shown in oblique view, are good subjects for the brush, especially for "dry" shading, the effect to be aimed at being that of the rail section of Fig. 136.

Fig. 136.



236. When some particular material is to be indicated, a flat tint of the proper technical color (see Art. 73) should be laid on with the brush, either before or after shading. When the latter is done with sepia it is probably safer to lay on the flat tint first.

A darker tint of the technical color should always be given to a cross-section. For blue-printing, a cross-section may be indicated in solid black.

WOOD. — RIVER-BEDS. — MASONRY, ETC.

237. While the engineering draughtsman is ordinarily so pressed for time as not to be able to give his work the highest finish, yet he ought to be able, when the occasion demands, to obtain both natural and artistic effects; and to conduce to that end the writer has taken pains to illustrate a number of ways of representing the materials of construction. Although nearly all of them may be — and in the cuts are — represented in black and white (with the exception of the wood-graining on Plate II), yet colors, in combined brush and line work, are preferable. The student will, however, need considerable practice with pen and ink before it will be worth while to work on a tinted figure.

238. Ordinarily, in representing wood, the mere fact that it *is* wood is all that is intended to be indicated. This may be done most simply by a series of irregular, approximately parallel lines, as on the rule in Fig. 17, page 12. Make no attempt, however, to have the grain *very* irregular. The natural unsteadiness of the hand in drawing a long line toward one continuously, will cause almost all the irregularity desired.

If a better effect is wanted, yet without color, the lines may be as in Fig. 107, which represents hard wood.

In graining, the draughtsman should make his lines *toward* himself, standing, so to speak, at the end of the plank upon which he is working.

The splintered end of a plank should be sharply toothed, in contradistinction to a metal or stone fracture, which is what might be called smoothly irregular.

239. An examination of any piece of wood on which the grain is at all marked will show that it is darker at the inner vertex of any marking than at the outer point. Although this difference is more readily produced with the brush, yet it may be shown in a satisfactory degree with the pen, by a series of after-touches.

240. If we fill the pen with a rather dark tint of the conventional color, draw the grain as in the figures just referred to and then overlay all with a medium flat wash of some properly chosen color, we get effects similar to those of Plate II.

On large timber-work the preliminary graining, as also the final wash, may be done altogether with the brush: as was the original of Fig. 9, Plate II.

End views of timbers and planks are represented by a series of concentric free-hand rings, the spacing of which increases with the distance from the heart; these are overlaid with a few radial strokes of darker tint. In ink alone the appearance is shown in Figs. 39 and 115.

241. The mixtures recommended for wood graining are something short of infinite in number: but with the addition of one or two colors to those listed in the draughtsman's outfit (Art. 56) one should be able to imitate the tint of any natural wood he might have in hand.

Delicate rather than glaring tints should be employed.

No hard-and-fast rule as to the proportions of the colors can be given. In this connection one recalls Sir Joshua Reynolds' reply—"With brains"—to the one who inquired how he mixed his paints.

Merely to indicate wood with a color and no graining use burnt sienna, the tint of Figs. 7, 8 and 10 of Plate II.

Drawing from the writer's experience and from the suggestions of various experimenters in this line the following hints are presented:—

In every case grain first, then overlay with the ground tint, which should always be much lighter than the color used for the grain. If possible have at hand a good specimen of the wood to be imitated.

Hard Pine: Grain—burnt umber with either carmine or crimson lake; for overlay add a little gamboge to the grain-tint diluted.

Soft Pine: Gamboge or yellow ochre with a small amount of burnt sienna.

Black Walnut: Grain—burnt umber and a very little dragon's blood; final overlay of modified tint of the same or with the addition of Payne's gray.

Oak: Grain—burnt sienna; for overlay, the same, with yellow ochre.

Chestnut: Grain—burnt umber and dragon's blood; overlay of the same, diluted, and with a large proportion of gamboge or light yellow added.

Spruce: Grain—burnt umber, medium; add yellow ochre for the overlay.

Mahogany: Grain—burnt sienna or umber with a small amount of dragon's blood; dilute and add light yellow for the overlay.

Rosewood: Grain—replace the dragon's blood of mahogany-grain by carmine, and for overlay dilute and add a little Prussian blue.

242. *River-beds* in black and white or in colors have been already treated in Art. 27, to which it is only necessary to add that such sections are usually made quite narrow and, preferably,—if in color—shaded out quite abruptly on the side opposite the water.

243. The sections of *masonry*, *concrete*, *brick*, *glass* and *vulcanite*, given on page 25 as pen and ink exercises, are again presented in Fig. 137 for reproduction in combined brush and line work. The appropriate color is indicated under each section.

Fig. 137.

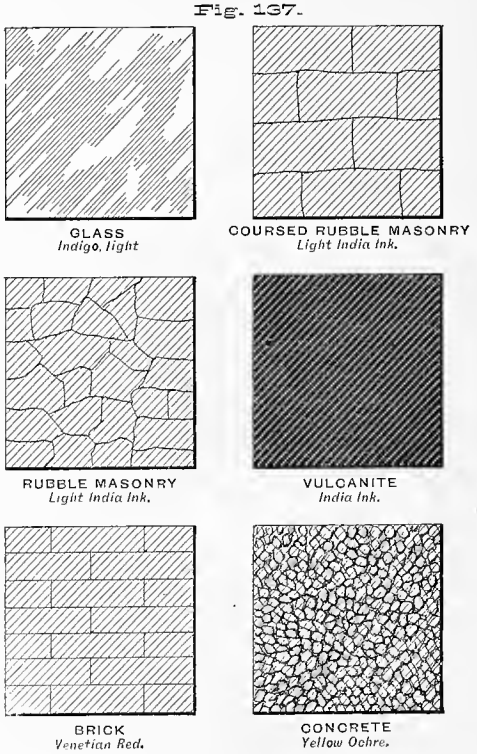
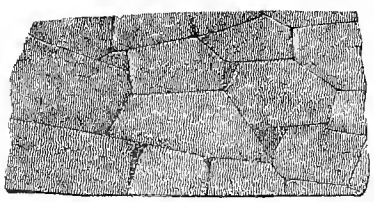


Fig. 138.

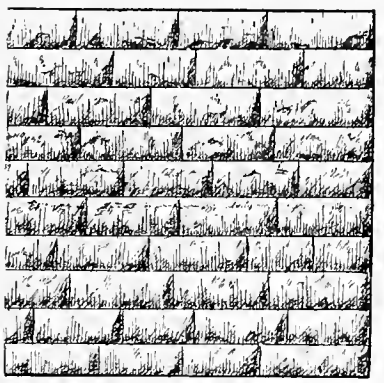


244. *Masonry* constructions are roughly divided into *rubble* and *ashlar*.

In *ashlar* masonry the bed-surfaces and the joints (edges) are shaped and dressed with great care so that the stones may not only be placed in regular layers or *courses*, but often fill exactly some predetermined place, as in arch-construction, in which case the determination of their forms and the derivation of the patterns for the stone-cutter involves the application of *Monge's Descriptive.*

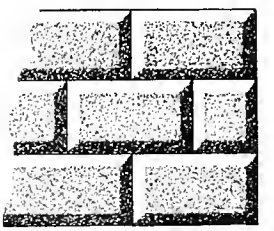
Rubble work, however, consists of constructions involving stones mainly "in the rough," but may be either coursed or uncoursed. Fig. 138 is a neat example of uncoursed though partially dressed or "hammered" rubble. In section, as shown in Fig. 137, it is merely necessary to rule section-lines over the boundaries of the stones—a remark applying equally to ashlar masonry.

Fig. 139.



The other examples in this chapter are of ashlar, mainly "quarry-faced," that is, with the front nearly as rough as when quarried. A beveled or "chamfered" ashlar is shown in Figs. 129 and 140, the latter shaded in what is probably the most effective way for small work, viz., with *dots*, the effect depending upon the *number*, not the *size* of the latter.

Fig. 140.



Only a careful examination of the *kind* and *position* of the lines in the other figures will disclose the secret of the variety in the effects produced. For the handsomest results with any of these figures the pen-work—whether dotting

Fig. 141.

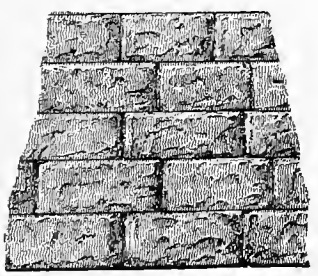
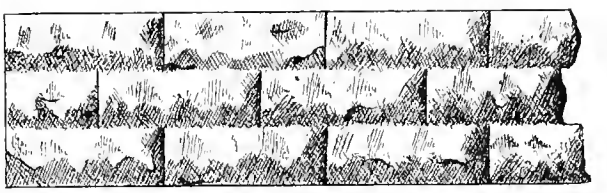


Fig. 142.



or "cross-hatching"—should be preceded by an undertone of either India ink, umber, Payne's gray, cobalt or Prussian blue, according to the kind of stone to be represented. For slate use a pale blue:

Fig. 143.

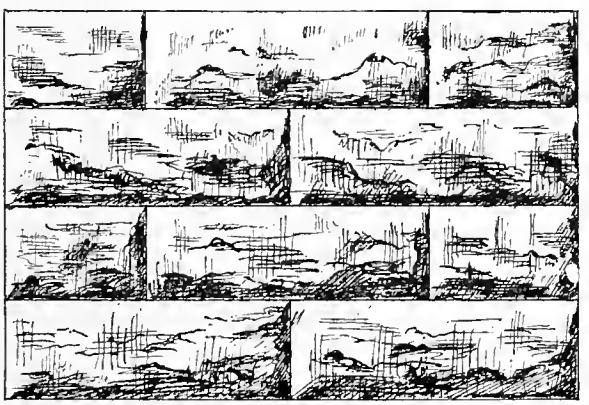
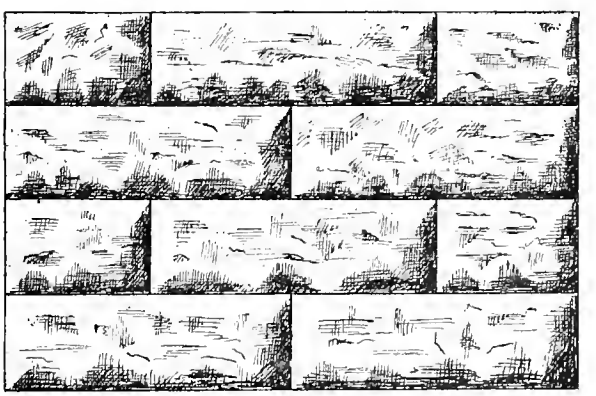


Fig. 144.



for brown free-stone either an umber or sepia; while for stone in general, *kind* immaterial, use India ink.

CHAPTER VII.

FREE-HAND AND MECHANICAL LETTERING.—PROPORTIONING OF TITLES.

245. Practice in lettering forms an essential part of the elementary work of a draughtsman. Every drawing has to have its title, and the general effect of the result as a whole depends largely upon the quality of the lettering.

Other things being equal, the expert and rapid draughtsman in this line has a great advantage over one who can do it but slowly. For this reason free-hand lettering is at a high premium, and the beginner should, therefore, aim not only to have his letters *correctly formed* and *properly spaced*, but, as far as possible, to do without mechanical aids in their construction. When under great pressure as to time it is, however, perfectly legitimate to employ some of the mechanical expedients used in large establishments as "short cuts" and labor-savers. Among these the principal are "tracing" and the use of rubber types.

246. To *trace* a title one must have at hand complete printed alphabets *of the size of type required*. Placing a piece of tracing-paper over the letter wanted, it is traced with a hard pencil, the paper then slipped along to the next letter needed and the process repeated until the words desired have been outlined. The title is then transferred to the drawing by first running over the lines on the back of the tracing-paper with a soft pencil, after which it is only necessary to re-trace the letters with a hard pencil, on the face of the transfer-paper, to find their outlines faintly yet sufficiently indicated on the paper underneath.

Carbon paper may also be used for transferring.

247. The process just described would be of little service to a ready free-hand draughtsman, but with the use of *rubber types*, for the words most frequently recurring in the titles, a merely average worker may easily get results which—in point of time—cannot be exceeded by any other method. When employing such types either of the following ways may be adopted: (a) a light impression may be made with the aniline ink ordinarily used on the pads, and the outlines then followed and the "filling in" done either with a writing-pen or fine-pointed sable-hair brush; or (b) the impression may be made after moistening the types on a pad that has been thoroughly wet with a light tint of India ink. The drawing-ink must then be immediately applied, free-hand, with a Falcon pen or sable brush, before the type-impression can dry. The pen need only be passed down the middle of a line, as on the dampened surface of the paper the ink will spread instantly to the outlines.

248. The educated draughtsman should, however, be able not only to draw a legible title of the simple character required for shop-work and in which the foregoing expedients would be mainly serviceable, but be prepared also for work out of the ordinary line and, if need be, quite elaborate, as on a competitive drawing. Such knowledge can only be gained by careful observation of the forms of letters and considerable practice in their construction.

No rigid rules can be laid down as to choice of alphabets for the various possible cases. Common-sense and a natural regard for the "fitness of things" must be the determining factors.

Obviously rustic letters would be out of place on a geometrical drawing, and other incongruities will naturally suggest themselves. In addition to the hints in Art. 26 a few general principles and instructions may, however, be stated to the advantage of the beginner.

249. In the first place, a title should be symmetrical with respect to a vertical centre-line, a rule which may be violated but rarely and then, usually, when the title is to be somewhat fancy in design, as for a magazine cover.

Elementary Plates
in
MECHANICAL DRAWING
drawn by Cortlandt Van Cortear at the
LEADING TECHNICAL SCHOOL
Jan.—June, 3001.

250. If it be a *complete* as distinguished from a *partial* or *sub*-title it will answer the following questions which would naturally arise in the mind of the examiner:—

What is it?—Where done?—By whom?—When?—On what scale?

In answering these questions the relative valuation and importance of the lines are expressed by the *sizes* and *kinds* of type chosen. This is a point requiring most careful consideration, as the final effect depends largely upon a proper balancing of values.

DETAIL DRAWINGS
OF
PERFECTION SUSPENSION BRIDGE
designed by
Goodwin, Mackenzie and Cartwright
MINNEAPOLIS, MINN.
Scale 4 Ft. = 1 In. June 16, 2900. Jose Martinez, Del.

251. The "By whom?" may cover two possibilities. In the case of a set of drawings made in a scientific school it would refer to the *draughtsman* and his name might properly have considerably greater prominence than in any other case. The upper title is illustrative of this point and at the same time suggestive of a good general arrangement for such a case.

Ordinarily the "By whom?" will refer to the designer, and the draughtsman's name ought to be comparatively inconspicuous, while the name of the designer should be given a fair degree of prominence. This, and other important points to be mentioned, are illustrated in the preceding

arrangement, printed, like the upper title, from types of which complete alphabets will be found in the Appendix.

252. The abbreviation *Del.*, following the draughtsman's name, is for *Delineavit—He drew it*—and does not indicate what the visitor at the exhibition supposed, that all good draughtsmen hail from Delaware.

253. The best designed titles are either in the form of two truncated pyramids having the most important line as their common base, or else elliptical in shape.

254. The use of capitals throughout a line depends upon the style of type. It gives a most unsatisfactory result if they are of irregular outline, as is amply evidenced by the words

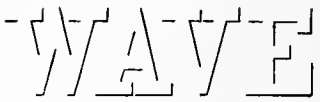
MECHANICAL DRAWING,

each letter of which is exquisite in form but the combination almost illegible. Contrast them with the same style, but in capitals and small letters:—

Mechanical Drawing.

255. As to *spacing*, the visible white spaces between the letters should be as nearly the same as possible. In this feature, as in others, the draughtsman can get much more pleasing results than the printer, since the latter has each letter on a separate piece of metal, and can not adjust his space to any particular combination of letters, such as F A, L V, W A or A V, where a better effect would be obtained by placing the lower part of one letter under the upper part of the next. This is illustrated in Fig. 146, which may be contrasted with the printer's best spacing of the A and W in the word "Drawings" of the last title.

Fig. 146.



256. The *amount of space* between letters will depend upon the length of line that the word or words must make. If an important word has few letters they should be "spaced out" and the letters themselves of the "extended" kind, i. e., broader than their height. The following word will illustrate. The characteristic feature of this type, viz., heavy horizontals and light verticals, is common to all the variations of a fundamental form frequently called *Italian Print*.

BRIDGE.

When, on the other hand, many letters must be crowded into a small space, a "condensed" style of letter must be adopted, of which the following is an example:—

Pennsylvania Railroad.

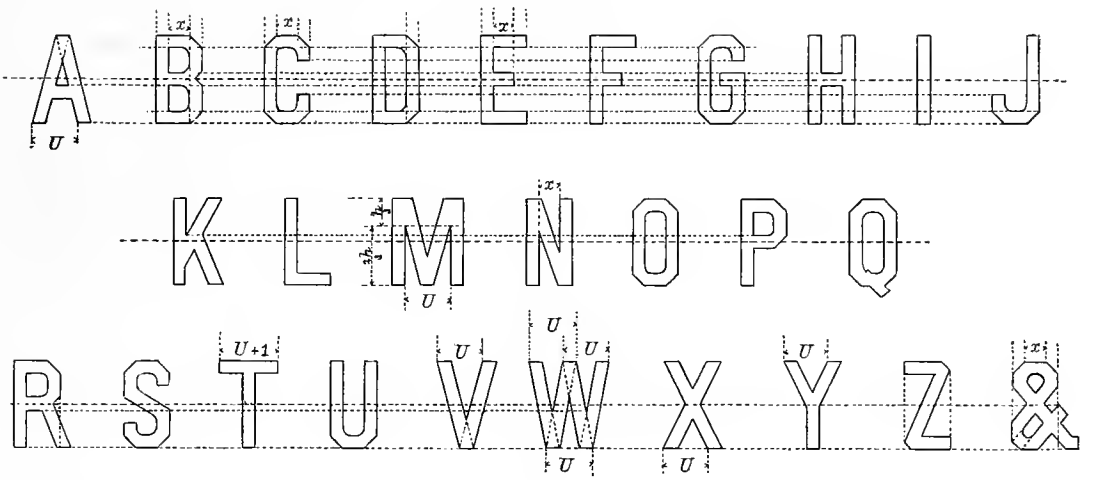
257. While the varieties of letters are very numerous, yet they are all but changes rung on a few fundamental or basal forms, the most elementary of which is the

GOTHIC, ALSO CALLED HALF-BLOCK.

Letters like B, O, etc., which have, usually, either few straight parts or none at all, may, for the sake of variety as also for convenience of construction, be made partially or wholly angular; in the latter case the form is called *Geometric Gothic* by some type manufacturers. It is especially appropriate for work exclusively mechanical.

The following complete Gothic alphabet is so constructed that whether designed in its "condensed" or "extended" form the proper proportions may be easily preserved.

Fig. 147.



Taking all the solid parts of the letters at the same width as the I, we will find any letter *of average width*, as U, to be twice that unit, plus the opening between the uprights, which last—being indeterminate—we may call x, making it small for a "condensed" letter and broad as need be for an "extended" form.

The word *march* would foot up 5 U + 3, disregarding—as we would invariably—the amount the foot of the R projects beyond the main right-hand outline of the letter. In terms of x this makes $5x + 13$, as $U = x + 2$. Allowing spaces of 1½ unit width between letters adds 5 to the above, making $5x + 18$ for the total length in terms of the I. Assuming x equal to twice the unit we would have the whole word equal to twenty-eight units; and if it were to extend seven inches the width of the solid parts would therefore be one-quarter of an inch.

Where the width of a letter is not indicated it is assumed to be that of the U. The W is equal to 2 U — 1. This relation, however, does not hold good in all alphabets.

The angular corners are drawn usually with the 45° triangle.

The guide-lines show what points of the various letters are to be found on the same level, and should be but faintly pencilled.

As remarked in Art. 26, the extended form of Gothic is one of the best for dimensioning and lettering *working drawings* and is rapidly coming into use by the profession.

258. The Full-Block letter next illustrated is easier to work with than the Gothic in the matter of preliminary estimate, as the width of each letter—in terms of unit squares—is evident at a glance.

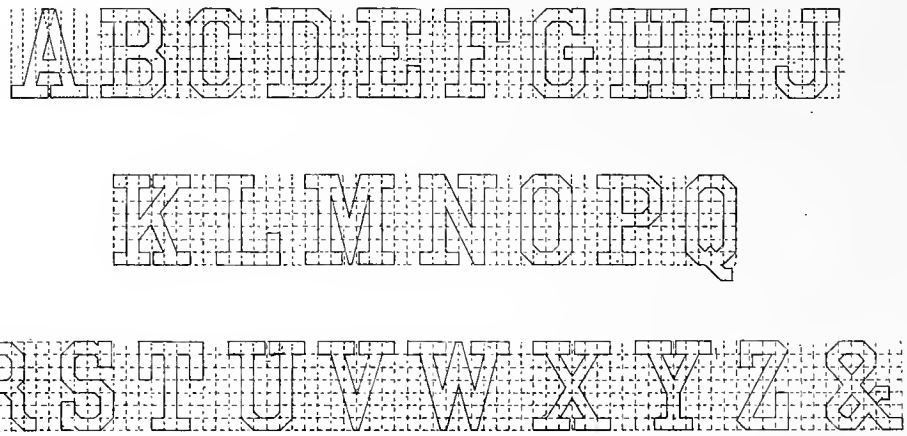
The same word *march* would foot up twenty-seven squares without allowing for spaces between letters. Calling the latter each *two* we would have thirty-five squares for the same length as before (seven inches) making one-fifth of an inch for the width of the solid parts. For convenience the widths of the various letters are summarized:

I = 3; C, G, O, Q, S, Z = 4; A, B, D, E, F, J, P, R, T, & = 5; H, K, N, U, V, X, Y = 6; M = 7; W = 8.

259. In case the preliminary figuring were only approximate and there were but two words in the line, as, for example, *Mechanical Drawing*, a safe method of working would be to make a fair allowance for the space between the words, begin the first word at the calculated distance to the left of the vertical centre-line, complete it, then work the second word backward, beginning with the

G as far to the right of the reference line as the M was to the left. On completing the second word any difference between the actual and the estimated length of the words, due to over- or under-width of such letters as M, W and I, will be merged into the space between the words.

Fig. 148.



With three words in a line the same method might be adopted, the middle word being easily placed half way between the others which, by the construction, would begin and terminate where they should.

260. Note particularly that the top of a B is always slightly smaller than the bottom; similarly with the S. This is made necessary by the fact that the eye seems to exaggerate the upper half of a letter. To get an idea of the amount of difference allowable compare the following equal letters printed from Roman type, condensed. Although not so important in the E still some difference between top and bottom may to advantage be made. Another refinement is the location of the horizontal cross-bar of an A slightly below the middle of the latter.

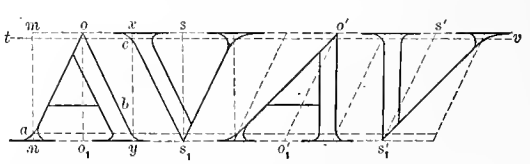
Fig. 149.

SS

261. While vertical letters are most frequently used yet no handsomer effect can be obtained than by a well-executed inclined letter. The angle of inclination should be about 70°.

Beginners usually fail sadly in their first attempt with the A and V, one of whose sides they give the same slant as the upright of the other letters. In point of fact, however, it is the imaginary (though, in the construction, pencilled) *centre-line* which should have that inclination. See Fig. 150.

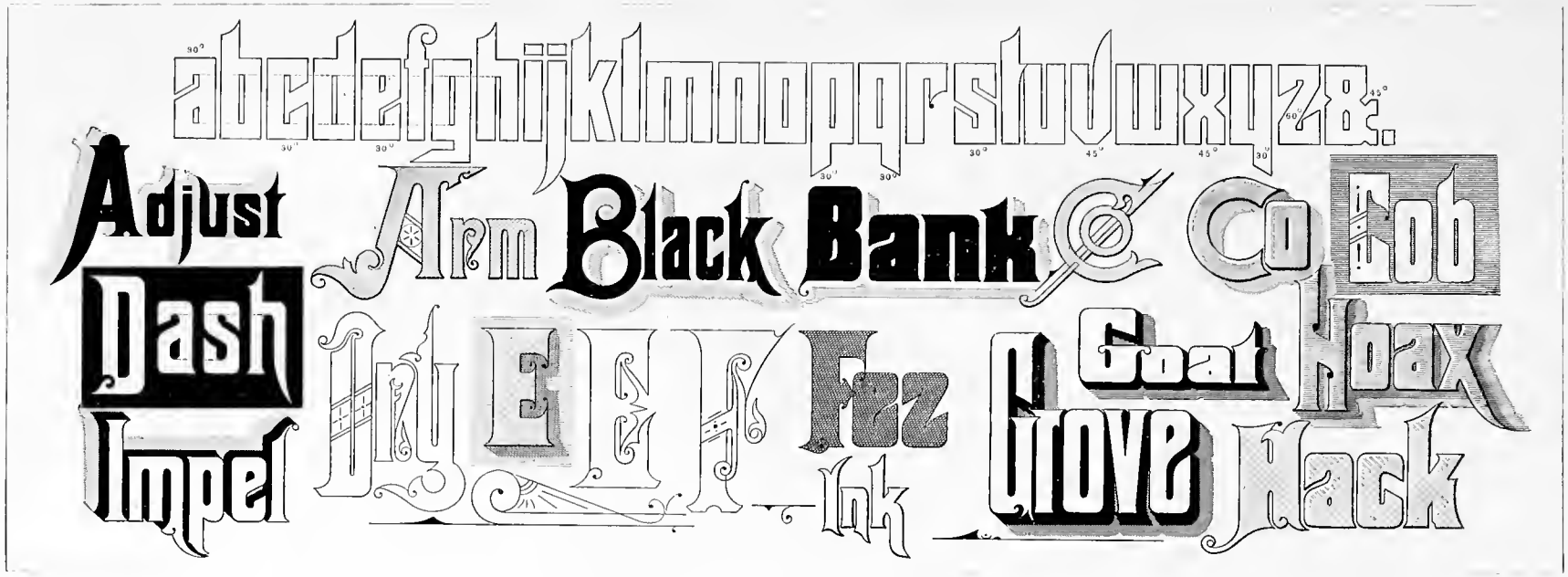
Fig. 150.



In these forms—the Roman and Italic Roman—the union of the light horizontals or "seriffs" with the other parts is in general effected by means of fine arcs, called "fillets," drawn free-hand. On many letters of this alphabet *some* lines will, however, meet at an angle, and only a careful examination of good models will enable one to construct correct forms. Upon the size of the fillets the appearance of the letter mainly depends, as will be seen by a glance at Fig. 151, which reproduces, exactly, the N of each of two leading alphabet books. If the fillets round out to the end of the spur of the letter a coarse and bulky appearance is evidently the result; while a fine curve, leaving the straight horizontals projecting beyond them, gives the finish desired. This is further illustrated by alphabet No. 23 of the Appendix, a type which for clearness and elegance is a triumph of the founder's art. If preferred, the D and R may be finished at the top like the P.

Fig. 151.

NN



RAILROAD TYPE.

262. The Roman alphabet and the Italic Roman are much used in topographical work, and while testing the draughtsman's skill to the utmost amply reward him for his labor.

A text-book devoted entirely to the Roman alphabet is in the market, and in some works on topographical drawing very elaborate tables of proportions for the letters are presented; these answer admirably for the construction of a standard alphabet, but in practice the proportions of the model would be preserved by the draughtsman no more closely than his eye could secure. Except when more than one-fifth of an inch in height these letters should be drawn entirely free-hand.

263. When a line of a title is curved no change is made in the *forms* of the letters; but if of a *vertical*, as distinguished from a slanting or *italic* type, the centre-line of each letter should, if produced, pass through the centre of the curve.

Italic letters, when arranged on a curve, should have their centre-lines inclined at the same angle to the normal (or radius) of the curve as they ordinarily make with the vertical.

264. An alphabet which gives a most satisfactory appearance, yet can be constructed with great rapidity, is what we may call the "Railroad" type, since the public has become familiar with it mainly from its frequent use in railroad advertisements.

The fundamental forms of the small letters, with the essential construction lines, are given in rectangular outline in the complete alphabet on the preceding page, with various modifications thereof in the words below them, showing a large number of possible effects.

At least one plain and fancy capital of each letter is also to be found on the same page, with in some instances a still larger range of choice.

No handsomer effects are obtainable than with this alphabet when brush tints are employed for the undertone and shadows.

265. For rapid lettering on tracing-cloth, Bristol board or any smooth-surfaced paper a style long used abroad and increasing in favor in this country is that known as *Round Writing*, illustrated by Fig. 152, and for which a special text-book and pens have been prepared by F. Soennecken. The pens are stubs of various widths, cut off obliquely, and when in use should not, as ordinarily, be dipped into the ink, but the latter should be inserted by means of another pen—between the *top* of the Soennecken pen and the brass "feeder" that is usually slipped over it to regulate the flow.

Fig. 152.

Round Writing

The Soennecken Round Writing Pens are also by far the best for lettering in *Old English*, *German Text* and kindred types.

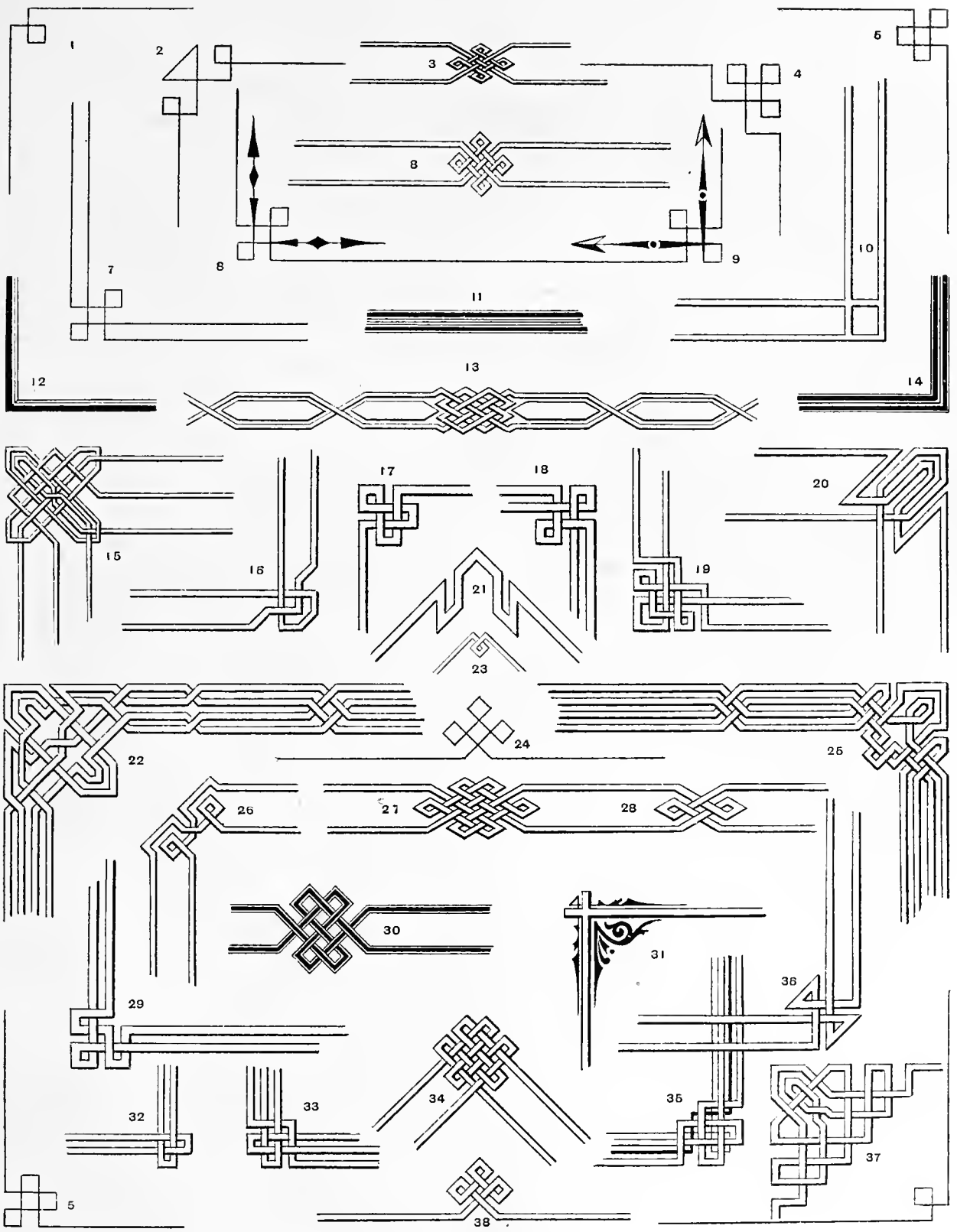
Fig. 153.

Elementary Plates
in
Mechanical Drawing

The improvement due to the addition of a few straight lines to an ordinary title will become evident by comparing Figs. 153 and 154. The judicious use of "word ornaments" such as those of alphabets 33, 42, 49 and of other types in the Appendix, will greatly enhance the appearance of a title, without materially increasing the time expended on it. This is illustrated in the lower title on page 89.

Fig. 154.

Elementary Plates
in
Mechanical Drawing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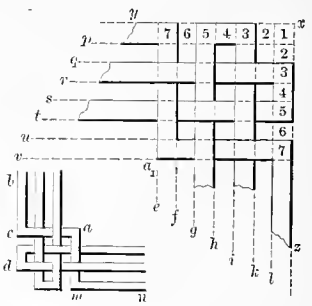
266. Another effective adjunct to a map or other drawing is a neat border. It should be strictly in keeping with the drawing, both as to character and simplicity.

On page 95 a large number of corner designs and borders are presented. The principle of their construction is illustrated by Fig. 155, in which the larger design shows the necessary preliminary lines and the smaller the completed corner. It is evident in this, as in all cases of interlaced designs, that we must lay off each way from the corner as many squares as there are bands and spaces, and make a network of squares—or of rhombi, if the angles are acute—by construction-lines through the points of division.

267. The usual rule as to shade lines applies equally to these designs, thus: Following any band or pair of lines making the turns as one piece, if it runs *horizontally* the *lower* line is the heavier, while in a *vertical* pair the *right-hand* line is the shade line. This is on the assumption that the light is coming in the direction usually assumed for mechanical drawings, i. e., descending diagonally from left to right.

Fig. 155.

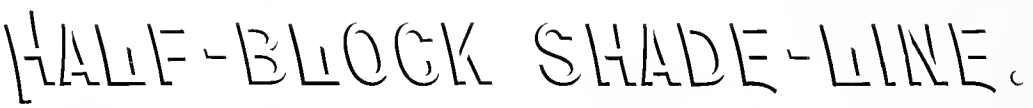


In case a pair of lines runs obliquely the shade lines may be determined by a study of their location on the designs of the plate of borders.

It need hardly be said that on any drawing and its title the light should be supposed to come from *but one direction throughout* and not be shifted; and the shade lines should be located accordingly. This rule is always imperative.

In drawing for scientific illustration or in art work it is allowable to depart from the usual strictly conventional direction of light if a better effect can thereby be secured.

268. A striking letter can be made by drawing the shade line only, as in Fig. 146, page 90, which we may call "Full-Block Shade-Line," being based upon the alphabet of Fig. 148, page 92, as to construction. Owing to its having more projecting parts it gives a much handsomer effect than the



The student will notice that the light comes from different directions in the two examples.

These forms are to the ordinary fully-outlined letters what art work of the "impressionist" school is to the extremely detailed and painstaking work of many; what is actually seen suggests an equal amount not on the paper or canvas.

269. While a teacher of draughting may well have on hand, as reference works for his class, such elaborate books on lettering as Prang's and Becker's, yet they will be found of only occasional service, their designs being as a rule more highly ornate than any but the specialist would dare undertake, and mainly of a character unsuitable for the usual work of the engineering or architectural draughtsman, whose needs were especially in mind when selecting types for this work.

The Appendix affords a large range of choice among the handsomest forms recently designed by the leading type manufacturers, also containing the best among earlier types; and with the "Railroad," Full-Block and Half-Block alphabets of this chapter, proportioned and drawn by the writer, supplies the student with a practical "stock-in-trade" that will probably require but little, if any, supplementing.

CHAPTER VIII.

BLUE-PRINT AND OTHER COPYING PROCESSES.—METHODS OF ILLUSTRATION.

270. While in a draughting office the process described below is, at present, the only method of copying drawings with which it is *absolutely essential* that the draughtsman should be thoroughly acquainted, he may, nevertheless, find it to his advantage to know how to prepare drawings for reproduction by some of the other methods in most general use. He ought also to be able to recognize, usually, by a glance at an illustration, the method by which it was obtained. Some brief hints on these points are therefore introduced.

This is, obviously, however, not the place to give full particulars as to all these processes, even were the methods of manipulation not, in some cases, still "trade secrets"; but all important details concerning them, that have become common property, may be obtained from the following valuable works: *Modern Heliographic Processes*,* by Ernst Lietze; *Photo-Engraving, Etching and Lithography*,† by W. T. Wilkinson; and *Modern Reproductive Graphic Processes*,* by Jas. S. Pettit.

THE BLUE-PRINT PROCESS.

271. By means of this process, invented by Sir John Herschel, any number of copies of a drawing can be made, in white lines on a blue ground. In Arts. 43 and 45 some hints will be found as to the relative merits of tracing-cloth and "Bond" paper, for the original drawing.

A sheet of paper may be sensitized to the action of light by coating its surface with a solution of red prussiate of potash (ferrocyanide of potassium) and a ferric salt. The chemical action of light upon this is the production of a ferrous salt from the ferric compound; this combines with the ferrocyanide to produce the final blue undertone of the sheet; while the portions of the paper, from which the light was intercepted by the lines of the drawing, become white after immersion in water.

The proportions in which the chemicals are to be mixed, are, apparently, a matter of indifference, so great is the disparity between the recipes of different writers; indeed, one successful draughtsman says: "Almost any proportion of chemicals will make blue-prints." Whichever recipe is adopted—and a considerable range of choice will be found in this chapter—the hints immediately following are of general application.

272. Any white paper will do for sensitizing that has a hard finish, like that of ledger paper, so as not to absorb the chemical solution.

To sensitize the paper dissolve the ferric salt and the ferrocyanide in water, separately, as they are then not sensitive to the action of light. The solutions should be *mixed* and *applied* to the paper only in a *dark room*.

Although there is the highest authority for "floating the paper to be sensitized for two minutes on the surface of the liquid" yet the best American practice is to apply the solution with a soft flat brush about four inches wide. The main object is to obtain an even coat, which may usually

* Published by the D. Van Nostrand Company, New York. † American Edition revised and published by Edward L. Wilson, New York.

be secured by a primary coat of horizontal strokes followed by an overlay of vertical strokes; the second coat applied before the first dries. If necessary, another coat of diagonal strokes may be given to secure evenness. The thicker the coating given the longer the time required in printing. A bowl or flat dish or plate will be found convenient for holding the small portion of the solution required for use at any one time. The chemicals should not get on the back of the sheet.

Each sheet, as coated, should be set in a dark place to dry, either "tacked to a board by two adjacent corners", or "hung on a rack or over a rod", or "placed in a drawer—one sheet in a drawer",—varying instructions, illustrating the quite general truth that there are usually several almost equally good ways of doing a thing.

273. To copy a drawing place the prepared paper, sensitized side up, on a drawing-board or printing-frame on which there has been fastened, *smoothly*, either a felt pad or canton flannel cloth. The drawing is then immediately placed over the first sheet, inked side up, and contact secured between the two by a large sheet of plate glass, placed over all.

Exposure in the direct rays of the sun for four or five minutes is usually sufficient. The progress of the chemical action can be observed by allowing a corner of the paper to project beyond the glass. It has a grayish hue when sufficiently exposed.

If the sun's rays are not direct, or if the day is cloudy, a proportionately longer time is required, running up in the latter case, from minutes into hours. Only experiment will show whether one's solution is "quick" or "slow;" or the time required by the degree of cloudiness.

A solution will print more quickly if the amount of water in it be increased or if more iron is used; but in the former case the print will not be as dark, while in the latter the results, as to whiteness of lines, are not so apt to be satisfactory.

Although fair results can be obtained with paper a month or more after it has been sensitized, yet they are far more satisfactory if the paper is prepared each time (and dried) just before using.

On taking the print out of the frame it should be immediately immersed and thoroughly washed in *cold water* for from three to ten minutes, after which it may be dried in either of the ways previously suggested.

If many prints are being made, the water should be frequently changed so as not to become charged with the solution.

274. The entire process, while exceedingly simple in theory, varies, as to its results, with the experience and judgment of the manipulator. To his choice the decision is left between the following standard recipes for preparing the sensitizing solution. The "parts" given are all by weight. In every case the potash should be pulverized, to facilitate its dissolving.

No. 1. (From *Le Génie Civil*).

Solution No. 1.	Red Prussiate of Potash	8 parts.
	Water	70 parts.
Solution No. 2.	Citrate of Iron and Ammonia	10 parts.
	Water	70 parts.

Filter the solutions separately, mix equal quantities and then filter again.

No. 2. (From U. S. Laboratory at Willett's Point).

Solution No. 1.	Double Citrate of Iron and Ammonia	1 ounce.
	Water	4 ounces.
Solution No. 2.	Red Prussiate of Potassium	1 ounce.
	Water	4 ounces.

No. 3. (Lietze's Method).

Stock Solution. { 5 ounces, avoirdupois, Red Prussiate of Potash.
32 fluid ounces .....Water.

"After the red prussiate of potash has been dissolved—which requires from one to two days—the liquid is filtered. This solution remains in good condition for a long time. Whenever it is required to sensitize paper, dissolve, for every two hundred and forty square feet of paper

{ 1 ounce, avoirdupois, Citrate of Iron and Ammonia,
4½ fluid ounces .....Water,

and mix this with an equal volume of the stock solution.

The reason for making a stock solution of the red prussiate of potash is, that it takes a considerable time to dissolve and because it must be filtered. There are many impurities in this chemical which can be removed by filtering. Without filtering, the solution will not look clear. The reason for making no stock solution of the ferric citrate of ammonia is that such solution soon becomes moldy and unfit for use. This ferric salt is brought into the market in a very pure state and does not need to be filtered after being dissolved. It dissolves very rapidly. In the solid form it may be preserved for an unlimited time, if kept in a well-stoppered bottle and protected against the moisture of the atmosphere. A solution of this salt, or a mixture of it with the solution of red prussiate of potash, will remain in a serviceable condition for a number of days, but it will spoil, sooner or later, according to atmospheric conditions. . . . Four ounces of sensitizing solution, for blue prints, are amply sufficient for coating one hundred square feet of paper, and cost about six cents."

For copying tracings in *blue* lines or *black*, on a white ground, one may either employ the recipes given in Lietze's and Pettit's works or obtain paper, already sensitized, from the leading dealers in draughtsmen's supplies. The latter course has become quite as economical, also, for the ordinary blue-print, as the preparing of one's own supply.

For copying a drawing in *any* desired color the following method, known as *Tilhet's*, is said to give good results: "The paper on which the copy is to appear is first dipped in a bath consisting of 30 parts of white soap, 30 parts of alum, 40 parts of English glue, 10 parts of albumen, 2 parts of glacial acetic acid, 10 parts of alcohol of 60°, and 500 parts of water. It is afterward put into a second bath, which contains 50 parts of burnt umber ground in alcohol, 20 parts of lampblack, 10 parts of English glue, and 10 parts of bichromate of potash in 500 parts of water. They are now sensitive to light, and must, therefore, be preserved in the dark. In preparing paper to make the positive print another bath is made just like the first one, except that lampblack is substituted for the burnt umber. To obtain colored positives the black is replaced by some red, blue or other pigment.

In making the copy the drawing to be copied is put in a photographic printing frame, and the negative paper laid on it, and then exposed in the usual manner. In clear weather an illumination of two minutes will suffice. After the exposure the negative is put in water to develop it, and the drawing will appear in white on a dark ground; in other words, it is a negative or reversed picture. The paper is then dried and a positive made from it by placing it on the glass of a printing-frame and laying the positive paper upon it and exposing as before. After placing the frame in the sun for two minutes the positive is taken out and put in water. The black dissolves off without the necessity of moving back and forth."

PHOTO-AND OTHER PROCESSES.

275. If a drawing is to be reproduced on a different scale from that of the original, some one of the processes which admits of the use of the camera is usually employed. Those of most importance to the draughtsman are (1) *wood engraving;* (2) the "wax process" or *cerography;* (3) *lithography*, and (4) the various methods in which the photographic negative is made on a film of gelatine which is then used *directly*—to print from, or *indirectly*—in obtaining a metal plate from which the impressions are taken.

In the first three named above the use of the camera is not invariably an element of the process.

All under the fourth head are essentially photo-processes and their already large number is constantly increasing. Among them may be mentioned *photogravure*, *collotype*, *phototype*, *autotype*, *photoglyph*, *albertype*, *heliotype*, and *heliogravure*.

WOOD ENGRAVING.

276. There is probably no process that surpasses the best work of skilled engravers on wood. This statement will be sustained by a glance at Figs. 14, 15, 20–24, 134, 136, and those illustrating mathematical surfaces, in the next chapter. Its expensiveness and the time required to make an illustration by this method are its only disadvantages.

Although the camera is often employed to transfer the drawing to the boxwood block in which the lines are to be cut, yet the original drawing is quite as frequently made *in reverse*, directly on the block, by a professional draughtsman who is supposed to have at his disposal either the object to be drawn or a photograph or drawing thereof. The outlines are pencilled on the block and the shades and shadows given in brush tints of India ink, re-enforced, in some cases, by the pencil, for the deepest shadows.

The "high lights" are brought out by Chinese white. A medium wash of the latter is also usually spread upon the block as a general preliminary to outlining and shading.

The task of the engraver is to reproduce faithfully the most delicate as well as the strongest effects obtained on the block with pencil and brush, cutting away all that is not to appear in black in the print. The finished block may then be used to print from directly, or an electrotype block can be obtained from it which will stand a large number of impressions much better than the wood.

CEROGRAPHY.

277. For map-making, illustrations of machinery, geometrical diagrams and all work mainly in straight lines or simple curves, and not involving too delicate gradations, the cerographic or "wax process" is much employed. For clearness it is scarcely surpassed by steel engraving. Figures 36, 90 and 107 are good specimens of the effects obtainable by this method. The successive steps in the process are (a) the laying of a thin, even coat of wax over a copper plate; (b) the transfer of the drawing to the surface of the wax, either by tracing or—more generally—by photography; (c) the re-drawing or rather the cutting of these lines in the wax, the stylus removing the latter to the surface of the copper; (d) the taking of an electrotype from the plate and wax, the deposit of copper filling in the lines from which the wax was removed.

Although in the preparation of the original drawing the lines may preferably be inked yet this is not absolutely necessary, provided a pencil of medium grade be employed.

Any letters desired on the final plate may be also pencilled in their proper places, as the engraver makes them on the wax with type.

A surface on which section-lining or cross-hatching is desired may have that fact indicated upon it in writing, the direction and number of lines to the inch being given. Such work is then done with a ruling machine.

Errors may readily be corrected, as the surface of the wax may be made smooth, for recutting, by passing a hot iron over it.

LITHOGRAPHY. — PHOTO-LITHOGRAPHY. — CHROMO-LITHOGRAPHY.

278. For the lithographic process a fine-grained, imported limestone is used. The drawing is made with a greasy ink—known as "lithographic"—upon a specially prepared paper, from which it is transferred, under pressure, to the surface of the stone. The un-inked parts of the stone are kept thoroughly moistened with water, which prevents the printer's ink (owing to the grease which the latter contains) from adhering to any portion except that from which the impressions are desired.

Photo-lithography is simply lithography, with the camera as an adjunct. The positive might be made directly upon the surface of the stone by coating the latter with a sensitizing solution; but, in general, for convenience, a sensitized gelatine film is exposed under the negative, and by subsequent treatment gives an image in relief which, after inking, can be transferred to the surface of the stone as in the ordinary process.

Chromo-lithography, or lithography in colors, has been a very expensive process owing to its requiring a separate stone for each color. Recent inventions render it probable that it will be much simplified and the expense correspondingly reduced. The details of manipulation are closely analogous to those for ink prints.

When colored plates are wanted, in which delicate gradations shall be indicated, chromo-lithography may preferably be adopted; although "half tones," with colored inks, give a scarcely less pleasing effect, as illustrated by Figs. 7–10, Plate II. But for simple line-work, in two or more colors, one may preferably employ either cerography or photo-engraving, each of which has not only an advantage, as to expense, over any lithographic process, but also this in addition—that the blocks can be used by any printer; whereas lithographing establishments necessarily not only prepare the stone but also do the printing.

PHOTO-ENGRAVING. — PHOTO-ZINCOGRAPHY.

279. In this popular and rapid process a sensitized solution is spread upon a smooth sheet of zinc and over this the photographic negative is placed. Where not acted on by the light the coating remains soluble and is washed away, exposing the metal, which is then further acted on by acids to give more *relief* to the remaining portions.

Except as described in Art. 281 this process is only adapted to inked work in lines or dots, which it reproduces faithfully, to the smallest detail. Among the best photo-engravings in this book are Figs. 12, 13, 50, 79 and 80.

280. The following instructions for the preparation of drawings, for reproduction by this process, are those of the American Society of Mechanical Engineers as to the illustration of papers by its members, and are, in general, such as all the engraving companies furnish on application.

"All lines, letters and figures must be *perfectly black* on a white ground. Blue prints are not available, and red figures and lines will not appear. The smoother the paper, and the blacker the ink, the better are the results. Tracing-cloth or paper answers very well, but rough paper—even

Whatman's—gives bad lines. India ink, ground or in solution, should be used; and the best lines are made on Bristol board, or its equivalent with an enameled surface. Brush work, in tint or grading, unfits a drawing for immediate use, since only line work can be photographed. Hatching for sections need not be completed in the originals, as it can be done easily by machine on the block. If draughtsmen will indicate their sections unmistakably, they will be properly lined, and tints and shadows will be similarly treated.

The best results may be expected by using an original twice the height and width of the proposed block. The reduction can be greater, provided care has been taken to have the lines far enough apart, so as not to mass them together. Lines in the plate may run from 70 to 100 to the inch, and there should be but half as many in a drawing which is to be reduced one half; other reductions will be in like proportion.

Draughtsmen may use photographic prints from the objects if they will go over with a carbon ink all the lines which they wish reproduced. The photographic color can be bleached away by flowing a solution of bi-chloride of mercury in alcohol over the print, leaving the pen lines only. Use half an ounce of the salt to a pint of alcohol.

Finally, lettering and figures are most satisfactorily printed from type. Draughtsmen's best efforts are usually thus excelled. Such letters and figures had therefore best be left in pencil on the drawings, so they will not photograph but may serve to show what type should be inserted."

To the above hints should be added a caution as to the use of the rubber. It is likely to diminish the intensity of lines already made and to affect their sharpness; also to make it more difficult to draw clear-cut lines wherever it has been used.

It may be remarked with regard to the foregoing instructions that they aim at securing that uniformity, as to general appearance, which is usually quite an object in illustration. But where the preservation of the individuality and general characteristics of one's work is of any importance whatever, the draughtsman is advised to letter his own drawings and in fact finish them entirely, himself, with, perhaps, the single exception of section-lining, which may be quickly done by means of *Day's Rapid Shading Mediums* or by other technical processes.

281. *Half Tones.* Photo-zincography may be employed for reproducing delicate gradations of light and shade, by breaking up the latter when making the photographic negative. The result is called a *half tone* and it is one of the favorite processes for high-grade illustration. Figs. 95 and 130 illustrate the effects it gives. On close inspection a series of fine dots in regular order will be noticed, so that no tone exists unbroken, but all have more or less white in them.

The methods of breaking up a tone are very numerous. The first patent dates back to 1852. The principle is practically the same in all, viz., between the object to be photographed and the plate on which the negative is to made there is interposed a "screen" or sheet of thin glass on which the desired mesh has been previously photographed.

In the making of the "screen" lies the main difference between the variously-named methods. In Meissenbach's method, by which Figs. 95 and 130 were made, a photograph is first taken, on the "screen," of a pane of clear glass in which a system of parallel lines—one hundred and fifty to the inch—has been cut with a diamond. The ruled glass is then turned at right angles to its first position and its lines photographed on the screen over the first set, the times of exposure differing slightly in the two cases, being generally about as 2 to 3.

This process is well adapted to the reproduction of "wash" or brush-tinted drawings, photographs, etc. The object to be represented, if small, may preferably be furnished to the engraving company and they will photograph it direct.

GELATINE FILM PHOTO-PROCESSES.

282. As stated in Art. 275, in which a few of the above processes are named, a gelatine film may be employed, either as an adjunct in a method resulting in a metal block, or to print from directly; in the latter case the prints must be made, on special paper, by the company preparing the film. In the composition and manipulation of the film lies the main difference between otherwise closely analogous processes. For any of them the company should be supplied with either the original object or a good drawing or photographic negative thereof.

Not to unduly prolong this chapter—which any intelligible distinction between the various methods would involve, yet to give an idea of the general principles of a gelatine process I conclude with the details of the preparation of a *heliotype* plate, given in the language of one company's circular. Figs. 1—5 of Plate II illustrate the effect obtained by it.

"Ordinary cooking gelatine forms the basis of the positive plate, the other ingredients being bichromate of potash and chrome alum. It is a peculiarity of gelatine, in its normal condition, that it will absorb *cold water*, and swell or expand under its influence, but that it will dissolve in *hot water*. In the preparation of the plate, therefore, the three ingredients just named, being combined in suitable proportions, are dissolved in hot water, and the solution is poured upon a level plate of glass or metal, and left there to dry. When dry it is about as thick as an ordinary sheet of parchment, and is stripped from the drying-plate, and placed in contact with the previously-prepared negative, and the two together are exposed to the light. The presence of the bichromate of potash renders the gelatine sheet sensitive to the action of light; and wherever light reaches it, the plate, which was at first gelatinous or absorbent of water, becomes leathery or waterproof. In other words, wherever light reaches the plate, it produces in it a change similar to that which tanning produces upon hides in converting them into leather. Now it must be understood that the negative is made up of transparent parts and opaque parts; the transparent parts admitting the passage of light through them, and the opaque parts excluding it. When the gelatine plate and the negative are placed in contact, they are exposed to light with the negative *uppermost*, so that the light acts through the translucent portions, and waterproofs the gelatine underneath them; while the opaque portions of the negative shield the gelatine underneath them from the light, and consequently those parts of the plate remain unaltered in character. The result is a thin, flexible sheet of gelatine, of which a portion is waterproofed, and the other portion is absorbent of water, the waterproofed portion being the image which we wish to reproduce. Now we all know the repulsion which exists between water and any form of *grease*. Printer's ink is merely grease united with a coloring-matter. It follows, that our gelatine sheet, having water applied to it, will absorb the water in its unchanged parts; and, if ink is then rolled over it, the ink will adhere only to the waterproofed or altered parts. This flexible sheet of gelatine, then, prepared as we have seen, and having had the image impressed upon it, becomes the *heliotype plate*, capable of being attached to the bed of an ordinary printing-press, and printed in the ordinary manner. Of course, such a sheet must have a solid base given to it, which will hold it firmly on the bed of the press while printing. This is accomplished by uniting it, under water, with a metallic plate, exhausting the air between the two surfaces, and attaching them by atmospheric pressure. The plate, with the printing surface of gelatine attached, is then placed on an ordinary platen printing-press. and inked up with ordinary ink. A mask of paper is used to secure white margins for the prints; and the impression is then made, and is ready for issue."

"*The study of Descriptive Geometry possesses an important philosophical peculiarity, quite independent of its high industrial utility. This is the advantage which it so pre-eminently offers in habituating the mind to consider very complicated geometrical combinations in space, and to follow with precision their continual correspondence with the figures which are actually traced—of thus exercising to the utmost, in the most certain and precise manner, that important faculty of the human mind which is properly called 'imagination,' and which consists, in its elementary and positive acceptation, in representing to ourselves, clearly and easily, a vast and variable collection of ideal objects, as if they were really before us. . . . . While it belongs to the geometry of the ancients by the character of its solutions, on the other hand it approaches the geometry of the moderns by the nature of the questions which compose it. These questions are in fact eminently remarkable for that generality which constitutes the true fundamental character of modern geometry; for the methods used are always conceived as applicable to any figures whatever, the peculiarity of each having only a purely secondary influence.*"

AUGUSTE COMTE: Cours de Philosophie Positive.

"*A mathematical problem may usually be attacked by what is termed in military parlance the method of 'systematic approach;' that is to say, its solution may be gradually felt for, even though the successive steps leading to that solution cannot be clearly foreseen. But a Descriptive Geometry problem must be seen through and through before it can be attempted. The entire scope of its conditions as well as each step toward its solution must be grasped by the imagination. It must be 'taken by assault.'*"

GEORGE SYDENHAM CLARKE, Captain, Royal Engineers.

CHAPTER IX.

ORTHOGRAPHIC PROJECTION UPON MUTUALLY PERPENDICULAR PLANES.

283. In this and the succeeding chapter, as also in nearly all of the latter part of this work, the principles of what has been generally known as *Descriptive Geometry* are either examined or applied.

In Art. 19—which, with Arts. 2, 3 and 14, should be reviewed at this point—reasons are given for calling this science *Monge's Descriptive.* Certain German writers call it *Monge's Orthogonal Projection.* The popular titles *Mechanical Drawing, Practical Solid Geometry, Orthographic Projection,* etc., are usually merely indicative of more or less restricted applications of Monge's Descriptive to some special industrial arts; and *working drawings* of bridge and roof trusses, machinery, masonry and other constructions, are simply accurately scaled and fully dimensioned projections made in accordance with its principles.

Monge's service to mathematics and graphical science, which, according to Chasles*, inaugurated the fifth epoch in geometrical history, consisted, not in inventing the method of representing objects by their projections—for with that the ancients were thoroughly familiar, but in perceiving and giving scientific form to the principles and theorems which were fundamental to the special solutions of a great number of graphical problems handed down through many centuries, and many of which had been the monopoly of the Freemasons. Emphasizing in this chapter the abstract principles of the subject, treating it as a pure science, and giving a mathematical outline of the field of its application, I leave for the next chapter its more practical aspect, including the modifications in vogue in the draughting offices of leading mechanical engineers.

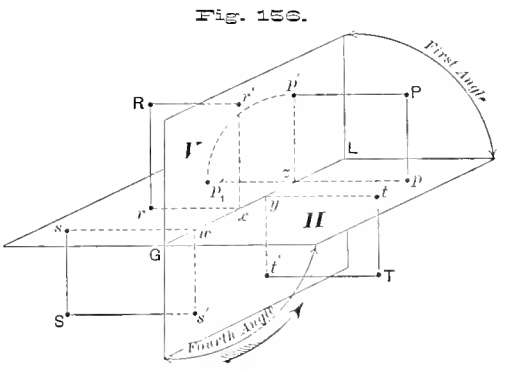
Statements without proof are given whenever their truth is reasonably self-evident.

FUNDAMENTAL PRINCIPLES.

284. The orthographic projection of a point on a plane is the foot of the perpendicular from the point to the plane.

In Fig. 156 the perpendiculars $P\,p'$ and $P\,p$ give the projections, p' and p, of the point P.

Fig. 156.



The planes of projection are shown in their space position; one, H, horizontal, the other, V, vertical.

The projection, p, on H, is called the *plan* or *horizontal projection* (h. p.) of P. The point p' is the *elevation* or *vertical projection* (v. p.) of P.

Projections on the vertical plane are denoted by small letters with a single accent or "prime." Projections on H are small letters unaccented.

A point may be named by its space-letter, the capital, or by its projections; thus, we may speak of the point P or of the point $p\,p'$.

* Aperçu Historique sur l'Origin et le Developpement des Methodes en Géométrie.

We shall call $P\,p$ the H-*projector* of P, since it gives the projection of P on H. Similarly, $P\,p'$ is the V-*projector* of P. A *projector-plane* is then the plane containing both projectors of a point, and is evidently perpendicular to both V and H by virtue of containing a line perpendicular to each.

285. V and H intersect in a line called the *ground line*, hereafter denoted by G. L. They make with each other four diedral angles.

The observer is always supposed to be in the *first angle*, viz., that which is above H and in front of V. We shall call it Q_1, or the *first quadrant*. Q_2 is then the second quadrant, back of V but above H. Q_3 is below Q_2, while Q_4 is immediately below the first angle.

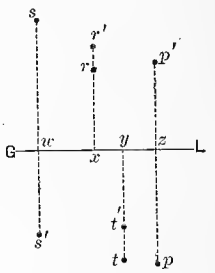
Points R and S, in the second and third quadrants respectively, have their elevations, r' and s', on opposite sides of G. L., while their plans, r and s, are on the back half of H.

The point T, in Q_1, has its v. p. at t', below G. L., and its h. p. at t, in front of G. L.

286. In making a drawing in the ordinary way, and not pictorially, we suppose the planes V and H brought into coincidence by revolution about G. L., the upper part of V uniting with the back part of H, while lower V and front H merge in one. The arrow shows such direction of revolution, after which any vertical projection, as p', is found at p'_1, on a line $p\,z$ perpendicular to G. L. and containing the plan p. This is inevitable, from the following consideration: Any point when revolved about an axis describes a circle whose centre is on the axis and whose plane is perpendicular to the axis; but as the projector-plane of P contains p'—the point to be revolved, and is perpendicular to G. L. (the axis) because perpendicular to both H and V, it must be the plane of rotation of p', which can therefore only come into H somewhere on $p\,z$ (produced).

In Fig. 157 we find Fig. 156 represented in the ordinary way. Only the projections of the points appear. V and H are, as usual, considered indefinite in extent, and their boundaries have disappeared. A projector-plane is shown only by the line, perpendicular to G. L., in which its intersections with V and H coincide after revolution.

Fig. 157.

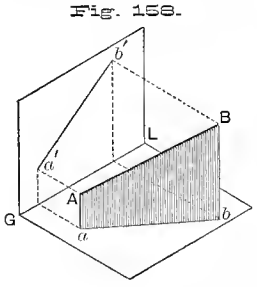


287. A point $(p\,p')$ in the *first* angle has its h. p. below G. L. and its v. p. above. For the *third* angle the reverse is the case, the *plan*, s, *above*, and the *elevation*, s', *below*. The second and fourth angles are also opposites, both projections, $r\,r'$, being above G. L. for the former, and both, $t\,t'$, below for the latter.

288. For any angle the actual distance of a point from H, as shown by any H-projector $P\,p$ (Fig. 156), is equal to $p'\,z$ (either figure)—the distance of the v. p. of the point from G. L. Similarly, the V-projector of a point, as $R\,r'$ (Fig. 156), showing the actual distance of a point from V, is equal to the distance of the h. p. of the point from G. L.

If one projection of a point is on G. L. the point is *in* a plane of projection. If in H, the *elevation* of the point will be on G. L.; similarly the *plan*, if the point lies in V.

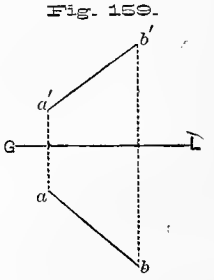
Fig. 158.



289. The projection of a line is the line containing the projections of all its points. The projection of a straight line will be a straight line; for its extremity-projectors, as $A\,a$ and $B\,b$ (Fig. 158), would determine a plane perpendicular to H and containing $A\,B$; in such plane all other H-projectors must lie, hence meet H in $a\,b$, which is straight because the intersection of two planes.

In Fig. 159 we see the line $A\,B$ of Fig. 158, orthographically represented.

Fig. 159.



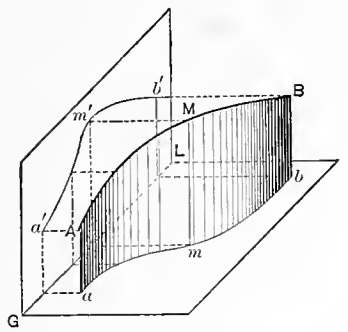
290. A plane containing the projectors of a straight line is called a *projecting plane* of the line. ($A\,B\,b\,a$, Fig. 158).

A V-*projecting plane* is the plane through a line and perpendicular to the vertical plane.

The H-*projecting plane* of a line is the vertical plane containing it.

The intersection of a surface by a given line or surface is called a *trace*. The trace of a *line* is a *point;* of a *surface* is a *line*. If on H it is called a *horizontal trace* (*h. t.*); on V, a *vertical trace*, abbreviated to *v. t.*

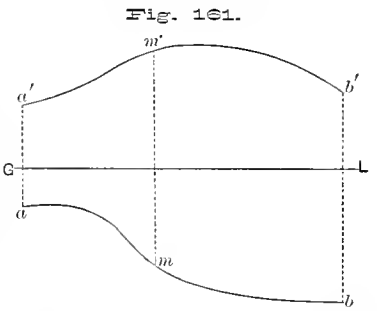
Fig. 160.



291. The projection of a *curve* is in general a curve, the *trace*—upon V or H—of the cylindrical surface* whose elements are the projectors of the points of the curve. (See Figs. 160 and 161).

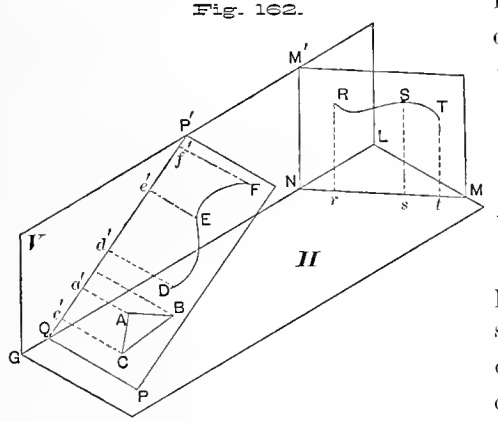
The projection of a plane curve will be equal and parallel to the original curve when the latter is parallel to the plane on which it is projected.

Fig. 161.



292. All lines, straight or curved, lying in a plane that is perpendicular to V, will be projected on V in the v. t. of the plane. A similar remark applies to the plans of lines lying in a *vertical* plane. Fig. 162 illustrates these statements. The plane $P'Q\,P$, being perpendicular to V—as shown by the h. t. ($P\,Q$) being perpendicular to G. L—all points in the plane, as A, B, C, D, E, F, will have their projections on V in the trace $P'Q$.

Fig. 162.

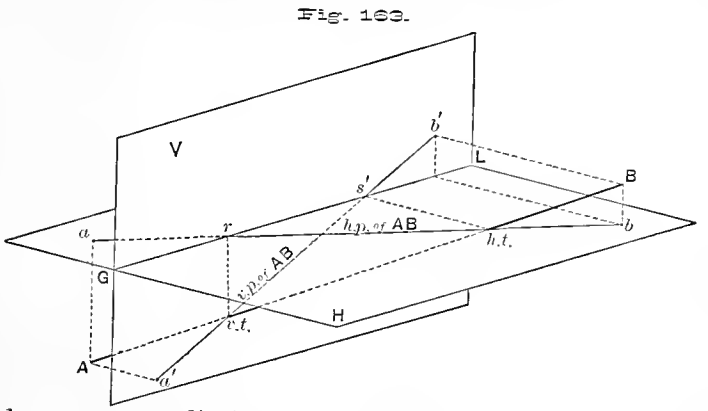


Plane $M'N\,M$ is vertical, since $M'N$—its *vertical* trace—is perpendicular to G. L; its h. t. ($N\,M$) therefore contains the h. p. of every point in the plane.

Since the triangle $A\,B\,C$ and the curve $D\,E\,F$ are not parallel to H or V their exact *size* and *shape* would not be shown by their projections; these could, however, be readily obtained by rotating their plane into H, about the trace $P\,Q$, or into V about $P'\,Q$. Such rotation, called *rabatment*†, is described in detail in Art 306.

293. From Fig. 163 it is evident that the h. t. of a line $A\,B$, must be at the intersection of the plan $a\,b$ by a perpendicular to G. L. from s', where the elevation $a'\,b'$ crosses G. L.

Fig. 163.



Similarly, the v. t. of the line is on its v. p., immediately below r, the intersection of G. L. by the plan $a\,b$; hence the rule: *To find the horizontal trace of a line* prolong the vertical projection until it meets the ground line; then draw a perpendicular to the plan of the line. An analogous construction gives the vertical trace of the line.

Fig. 164.

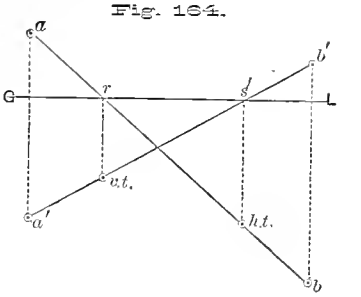


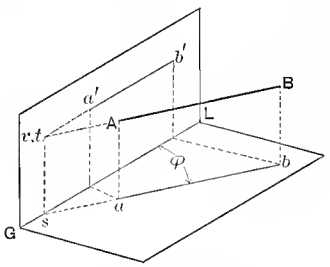
Fig. 164 shows, orthographically, the same projections and traces as Fig. 163.

* See Remark, Art 8.

† From the French *rabattement*.

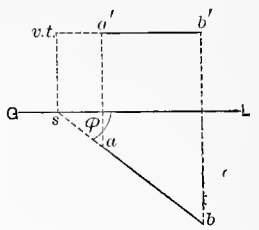
294. The inclination, θ, of the line to H, is that of the line to its plan $a\,b$; the angle ϕ, made with V, is that of the line to its elevation, $a'\,b'$.

Fig. 165.



295. Any *horizontal* line has a plan, $a\,b$ (Fig. 165), equal and parallel to itself. Its elevation, $a'\,b'$, is *parallel* to G. L., at the same distance from it as the line from H, and makes the same angle with V as its plan does with G. L. Such line can evidently have no h. t. Its v. t. would be found by the rule in the preceding article. In the extreme case of perpendicularity to V the v. p. would reduce to a point.

Fig. 166.



296. A line parallel to V but oblique to H has its h. p. parallel to G. L., makes with H the same angle that its v. p. does with G. L., and has its h. t. found by the usual rule. (Fig. 167.)

Fig. 167.

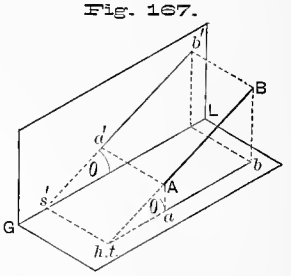
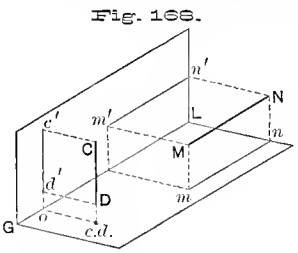


Fig. 168.



297. A vertical line has no v. t.; is projected on H in a point; has its v. p. parallel and equal to itself. (See $C\,D$, Fig. 168).

A line parallel to both V and H is parallel to their intersection, has no traces, and each projection equals the line. (See $M\,N$, Fig. 168.)

Fig. 169 shows, orthographically, the lines $A\,B$, $C\,D$ and $M\,N$ of the two preceding figures.

298. A *plane* is represented by its traces. Like H and V, any plane is considered indefinite in extent when drawn in the usual way; though our pictorial diagrams show them bounded, to add to the appearance.

Fig. 169.

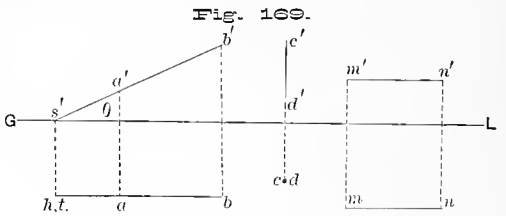
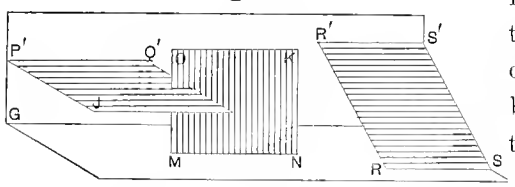


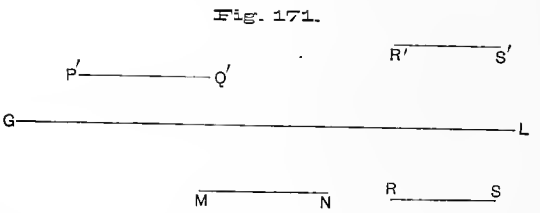
Fig. 170.



A *horizontal* plane has but one trace, that on V. A plane parallel to V, as $M\,N\,K\,O$, Fig. 170, has no vertical trace, and its horizontal trace $M\,N$ is parallel to G. L.

If parallel to G. L., while inclined to H and V, a plane has parallel traces. ($R\,S$, $R'\,S'$, Fig. 170). Fig. 171 shows the planes of Fig. 170, as usually represented.

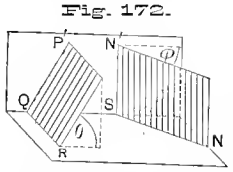
Fig. 171.



The traces of a plane not parallel to G. L. must meet at the same point on G. L.

299. We have stated that if a plane is perpendicular to a plane of projection, its trace on the other plane is perpendicular to G. L. This is the only case in which the angles between the ground line and the traces of a plane equal the dihedral angles made with H and V by the plane. That such equality exists may be thus demonstrated: The dihedral angle between two planes equals the plane angle between two lines, cut from the planes by a third plane perpendicular to both; plane $P'\,Q\,R$ (Fig. 172) is by hypothesis perpendicular to V; hence the ground line and $P'Q$ are the lines cut by plane V from two other planes to which it is perpendicular; and their angle θ is, therefore, the measure of the dihedral angle between H and $P'Q\,R$. The same course of reasoning may be applied to each trace of both planes.

Fig. 172.



300. In Fig. 173 we have a plane oblique to both H and V. Two important lines are drawn in it which we shall have frequent occasion to use.

Any line parallel to H is necessarily a horizontal line; but when also contained by a plane it is called a "horizontal" of the plane. It evidently must be parallel to the h. t. ($Q\,R$) of the plane.

Any line parallel to V can have that fact discovered by the parallelism of its plan to G. L.;

Fig. 173.

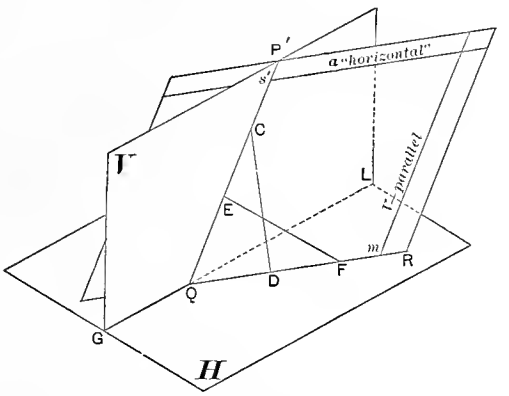
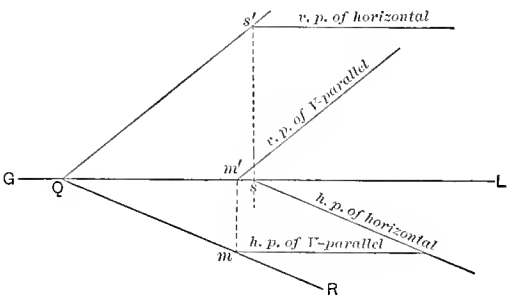


Fig. 174.



but when contained by a plane, we shall call it a *V-parallel* of the plane. Such line will evidently be parallel to the v. t. of the plane in which it lies.

The oblique lines $C\,D$ and $E\,F$, with those just described, illustrate the additional fact that the traces of lines in a plane will be found on the traces of the plane. This furnishes the following—and usual—method of determining a plane, i. e., by means of two lines known to lie in it: Prolong the lines until they meet the plane of projection; their H-traces joined give the h. t. of the plane. Similarly for the vertical trace of the plane.

Two intersecting lines or two parallel lines determine a plane. Three points not in a straight line, or a point and straight line may be reduced to either of the foregoing cases.

301. In any plane, a line making with H or V the same angle as the plane, is called a *line of declivity* of the plane.

Fig. 175.

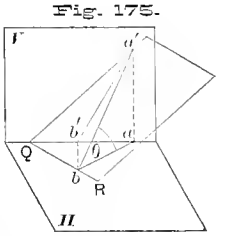
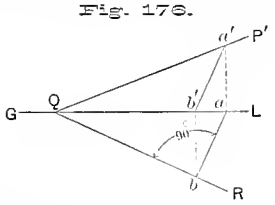


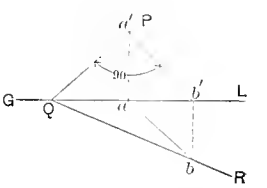
Fig. 175 shows a line of declivity with respect to H. Both the line and its plan must be perpendicular to the h. t. of the plane; for the inclination of a line to its plan is that of the line to H; and if, at the same time, the measure of a dihedral angle, such lines must, by elementary geometry, be perpendicular to the intersection of the planes. Fig. 176 shows Fig. 175 orthographically. Fig. 177 gives a line of declivity with respect to V.

Fig. 176.



302. Were the plane $P'Q\,R$ (Fig. 175) rotated on its line of declivity ($a\,b$, $a'\,b'$) it would make an increasing angle with H until it was perpendicular to it; that is, if a plane contains a line, the limits of its inclinations are 90° and that of the line.

Fig. 177.



On the other hand, the line of declivity might be turned *in the plane* until horizontal. The limits of the inclinations of lines in a plane are therefore 0° and that of the plane.

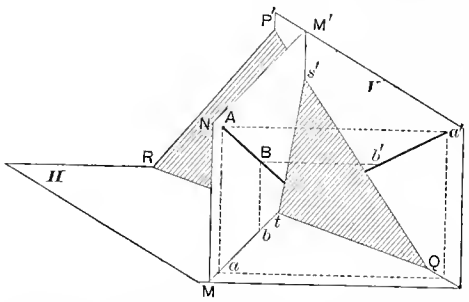
A plane may make 90° with H and be parallel to V, or vice versa; or it may be perpendicular to *both* H and V; the limits of the *sum* of its inclinations to H and V are thus 90° and 180°.

If parallel to G. L., but inclined to H and V, the sum of the inclinations of the plane is the lower limit, 90°.

If equally inclined to both H and V, but cutting G. L., the traces of a plane will meet the latter at equal angles.

303. If a right line, as $A\ B$ (Fig. 178), is perpendicular to a plane $P'\ Q\ R$, its projections will be perpendicular to the traces of the plane: for the plan of the line lies in the h. t. of its H-projecting plane; the latter plane is—from its definition—perpendicular to H, and also to the given plane by virtue of containing the given line; hence is perpendicular to the h. t. of the given plane, since such h. t. is the line of intersection of the latter with H.

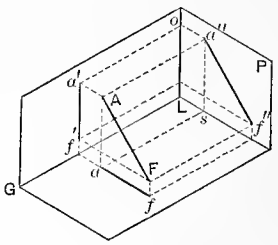
Fig. 178.



The same principles apply to the relation of the elevation of the line to the v. t. of the plane.

304. Any plane perpendicular to both H and V is called a *profile plane*. Such plane (P, Fig. 179) is used when *side* or *end* views of an object are to be projected. To bring V, H and P into one plane we suppose the latter first rotated into V about their line of intersection, $o\ L$, then both V and P about G. L. into H.

Fig. 179.

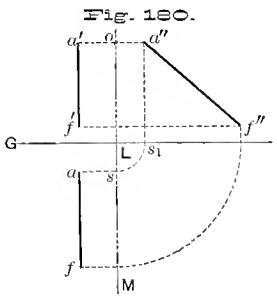


Projections on the profile plane are usually lettered with a double accent, the same as for any point revolved into or parallel to V.

When, as in Fig. 179 all the projectors of a line $A\ F$ lie in the same projector-plane, both projections of the line will be perpendicular to G. L. The most convenient method of dealing with such line is to project it upon a profile plane and revolve the latter into V; or the profile projector-plane through the line might be directly revolved into V, carrying the line with it. The former method is pictorially illustrated in Fig. 179. Orthographically, the operation is shown in Fig. 180.

In this, as in many other constructions, we make use of the following principles:

Fig. 180.

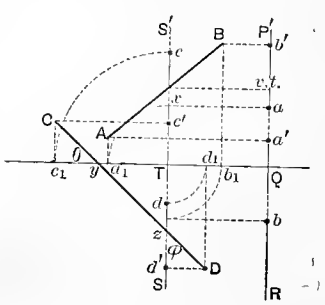


(a) all projections of one point on two or more *vertical* planes will be at the same height above H;

(b) if rotation occur about an axis that is perpendicular to H, each arc described about that axis by a point revolved, will be projected on H as an equal circular arc; similarly as to V, if the axis is perpendicular to it.

Since in Figs. 179 and 180 we rotate upon a *vertical* axis, a projector, as $A\ a''$, will be seen in $a\ s$, drawn through the plan of A and perpendicular to the h. t. of P. From s a circular arc, $s\ s_1$ (Fig. 180), from centre L, will be the plan of the arc described by a'' of Fig. 179. From s_1 a vertical to the level of a' gives a''. Similarly the projection f'' is obtained, which, joined with a'', gives $a''\ f''$, the profile view of the line $A\ F$.

Fig. 181.



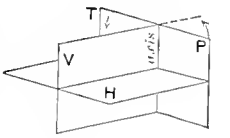
305. As far as our view of what is in the first angle is concerned, the rotation just described amounts, practically, to the turning of H and V through an angle of 90°, so that instead of facing V we see it "edgewise," as a line, $M\ o$; while H appears also as a line, $G\ L\ s_1$. We thus get an "end view" into the angle. All figures lying in profile planes are then seen in their true form.

In Fig. 181 let us start with the entire system of angles thus turned. The ground line appears as a point, T; H and V as lines; and two lines, $A\ B$ and $C\ D$—each of which lies in a profile plane—are shown in their true length and inclination.

Perpendiculars to H from A, B, C, and D, give their plans a_1, b_1, c_1 and d_1. V-projectors give d', c', etc., the heights of the elevations of the points.

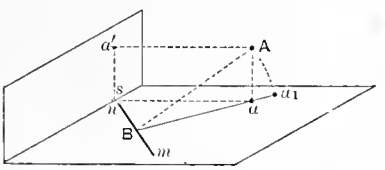
Revolving the whole system into its usual position, remembering, meanwhile, that the profile plane, P, turns on a *vertical axis*, (as in Fig. 182,) which divides P into parts which are on opposite sides of the axis both before and after revolution, we find c_1 at c; d_1 at d; a_1 at x. Assuming $S\,S'$ as the plane of the line $C\,D$, and that the plane of $A\,B$ is $R\,P'$, at a given distance $T\,Q$ from $S\,S'$, we find a' and b' on $R\,P'$ at the same level as A and B; while a is derived from x, and b from b_1 as shown. The elevations c' and d' on $S\,S'$ are on the level of C and D respectively.

Fig. 182.



306. The term *rabatment*, already employed to indicate the rotation of a *plane* about one of its traces until it comes into a plane of projection, is also used to denote the rotation of a *point* or *line* into H or V about an axis in such plane. Restoration to an original space-position, after revolving, will be called *counter-rabatment* or *counter-revolution*.

Fig. 183.

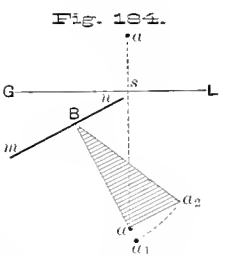


In Fig. 183 we have a_1 as the rabatment of A into H, about an axis $m\,n$. $B\,a_1$ is equal to $B\,A$—the actual distance of A from the axis, and which is evidently the hypothenuse of the triangle $A\,B\,a$, whose altitude is the H-projector of the point and whose base is the h. p. of the real distance.

Were the axis *parallel to* and not *in* H we would state the principle thus: In revolving a point about, and to the same level with, an *horizontal* axis it will be found on a perpendicular drawn through the h. p. of the point to the axis, and at a distance from the latter equal to the hypothenuse of a right-angled triangle whose altitude equals the *difference of level* of point and axis, and whose base is the h. p. of the real distance. Were the axis *in* or *parallel to* V, the *base* of the triangle constructed would be the *v. p.* of the desired distance, and the altitude would be—in the first case—the V-projector of the point, and—in the second case—the difference of distances of point and axis from V. In any case, the vertex of the right angle, in the triangle constructed, is the projection of the original point on the plane *in which* or *parallel to which* lies the axis.

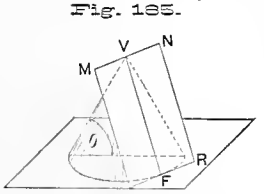
In usual position the foregoing construction would appear thus: with the point given by its projections, ($a\,a'$, Fig. 184), let fall a perpendicular through a to $m\,n$; prolong this indefinitely; make a the vertex of the right angle in a right triangle of base $B\,a$, altitude $a'\,s$; then the hypothenuse of such triangle, used as radius of arc $a_2\,a_1$ (centre B) gives a_1 as the revolved position desired.

Fig. 184.



307. In applying the foregoing principles in the following problems we shall frequently find it convenient to employ the right cone as an auxiliary surface. All the elements of such a cone are equally inclined to its base, and a tangent plane to the cone makes with the base the same angle as the elements. The element of tangency is a line of declivity of the plane with respect to the base of the cone.

Fig. 185.



If the base of the cone is on H the h. t. of a tangent plane will be tangent to the base of the cone; similarly for its v. t. were the base in V.

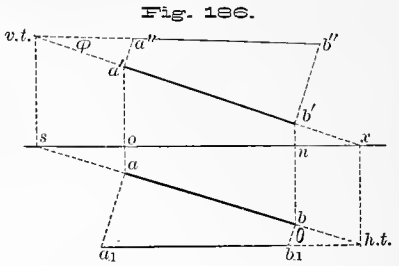
308. *Prob. 1. From the projections of a line to determine* (a) *its traces;* (b) *its actual length;* (c) *its inclinations, θ and ϕ, to* H *and* V *respectively.*

(a) *The traces of the given line*, when it is oblique to both H and V, as in Fig. 186, are found by the rule given in Art. 293.

(b) *The actual length* of a line may be found either (1) by *rabatment into* H or V, or (2) *by rotation until parallel to* H or V.

By the first method rabat the line on its plan $a\,b$ as an axis. It will show the true length on H in $a_1\,b_1$, the distance $a\,a_1$ equalling $a'\,o$ — the original height of the point A above H; similarly, $b\,b_1$ equals $b'\,n$. Notice that $a_1\,a$ and $b_1\,b$ must be perpendicular to the axis, and that each is the projection of a circular arc, described by the point revolved.

Fig. 186.

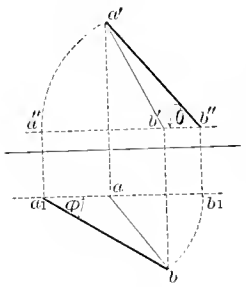


The h. t. being the point where the revolved line meets the axis of rotation, is common to both the rabatment, $a_1\,b_1$, and the space-position of the line.

If we make $a'\,a''$ and $b'\,b''$ perpendicular to $a'\,b'$ and equal to $a\,o$ and $b\,n$ respectively, we have in $a''\,b''$ the real length, shown on V.

By rotation till parallel to a plane of projection, as H, either extremity of the line may be brought to the level of the other, when the new plan will show the actual length. Thus, (Fig. 187), using the V-projector of $b\,b'$ as an axis, a' may be brought to a'', at the level of b', by an arc, centre b', radius $a'\,b'$. The circular arc $a'\,a''$ thus described has its h. p. in $a_1\,a$, the distance from V having been constant during the rotation, since the axis was perpendicular to V. In $a_1\,b$ we then have the real length sought. If we rotate the line on a vertical axis through a, until b reaches b_1, we will find the v. p. of the revolved point at b'', on its former level. The new projection, $a'\,b''$, is again the real length, now projected on V.

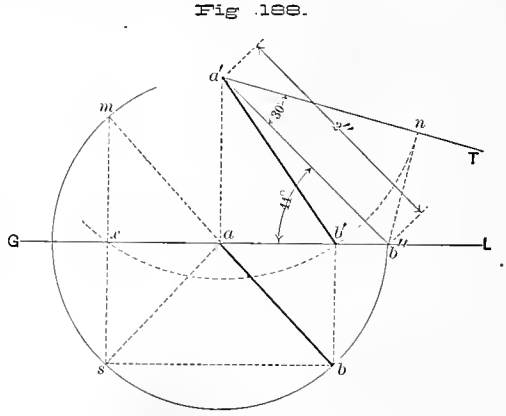
Fig. 187.



(c) *The inclinations, θ and ϕ, to* H *and* V *respectively.* Either of the foregoing constructions for showing the real length of a line solves also the problem as to inclination. Thus, in Fig. 186, the rabatted lines make with their axes of rabatment the angles sought. In Fig. 187 we have $a'\,b''$ inclined at the angle θ to H, while $b\,a_1$ makes with $a\,a_1$ the angle ϕ. *When the line lies in a profile plane*, the traces, length and inclination are found by means of the operation described in Art. 305 and illustrated by Fig. 181. In that figure, were $c\,d$ and $c'\,d'$ given, we would carry c and d about T as a centre to c_1 and d_1, whence perpendiculars to their former levels would give C and D; joining the latter we would have $C\,D$, whose v. t. is at z; h. t. (not shown) at a distance $T\,y$ above T; while θ and ϕ are seen in actual size at y and z.

309. *Prob. 2. To determine the projections of a line of given length, having given its angles, θ and ϕ, with* H *and* V *respectively.* If with the line we generate a vertical right cone, of base angle θ, four elements could be found on the cone, each of which would make the angle ϕ with V, and therefore fulfill all the conditions.

Fig. 188.



The sum of θ and ϕ can obviously not exceed 90°, and when equalling that limit there is but one solution and the line can then be contained by a profile plane. For data take *length of line*, 2″; $\theta = 44°$; $\phi = 30°$. From any point b'' on G. L., draw *in* V a line $b''\,a'$, of the given length and at the angle $\theta = 44°$ with G. L. The plan of this line is $a\,b''$, which use as the radius of the base of a semi-cone of vertical axis $a\,a'$.

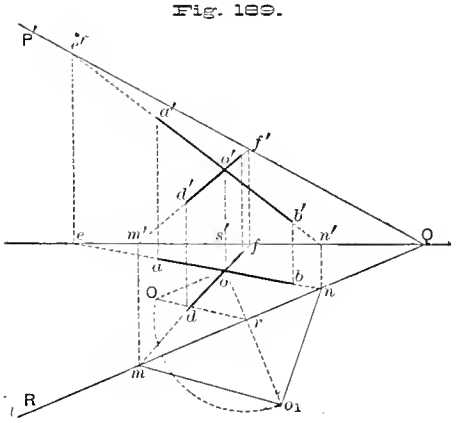
Remembering that the inclination of a line determines the length of its projection, we next ascertain how long the projection of a two-inch line will be when inclined 30° to a plane. Drawing $a'\,T$ at

30° to $a'\,b''$, and projecting b'' perpendicularly upon it at n, we find $a'\,n$ as the invariable length sought. Arc $n\,b'$ from centre a', gives $a'\,b'$ as the v. p. of an element of the cone, whence $a\,b$ follows, as the plan of the desired line.

As arc $n\,b'$, continued, would cut G. L. at x, whence s and m could be derived, we would find $s\,a$ and $m\,a$ as the plans of two more elements fulfilling the conditions. Also, in line with $b\,b'$, one more point (omitted to avoid confusing the solutions) could be found, on the rear of the cone.

310. *Prob. 3. To determine the plane containing* (a) *two intersecting lines;* (b) *two parallel lines.* In Fig. 189 let $o\,o'$ be the point of intersection of the lines $A\,B$ and $C\,D$. Prolong the lines and obtain their traces, as in Art. 293. $R\,Q$, the h. t. of the plane, is the line connecting m and n, the horizontal traces of the lines. Similarly, $P'\,Q$ passes through the vertical traces e' and f'.

Fig. 189.



(b). *Parallel lines* have parallel projections, and the traces of their plane connect like traces of the lines.

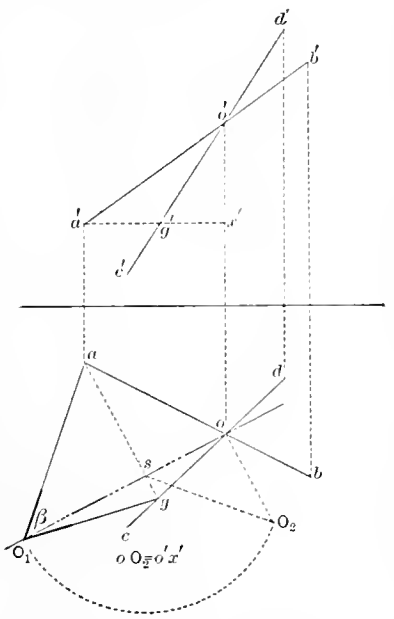
311. *Prob. 4. To show the actual size of the angle between two lines.*

If the actual angle is 90° it will be projected as such when one or both of its sides are parallel to the plane of projection: for, if *both are parallel*, the traces of the projecting planes—in which lie the projections of the lines—will evidently be perpendicular to each other; if either side of the angle in space be then rotated about the other side as an axis, it will turn in its previous projector-plane, and its projection will still fall upon the same trace as before.

In general, to show the true size of any angle, rotate its plane either *into* or *parallel to* H or V.

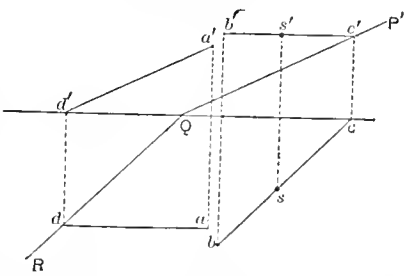
In Fig. 189 the angle whose vertex is $o\,o'$ is shown by obtaining the plane $R\,Q\,P'$ of its sides, then rabatting about $R\,Q$ into H. The vertex reaches o_1 after describing an arc whose plane is perpendicular to $R\,Q$ and which is projected in $o\,o_1$. The actual space-distance of O from r—the point on the axis, about which it turns—is the hypothenuse of a right triangle whose altitude $o\,O = o'\,s'$ and whose base is $o\,r$. Joining o_1 with m and n—the intersections of the axis by the lines revolved—gives $m\,o_1 n$, the angle sought.

Fig. 190.



To obtain the angle without having the traces of its plane we may use as an axis the line connecting points—one on each line—and equidistant from a plane of projection. Thus, in Fig. 190, we find a' and g' at the same level, and $a\,g$ for the plan of the line connecting them; then o rotates to O_1 about $a\,g$, giving $a\,O_1\,g$ for the desired angle.

Fig. 191.



312. *Prob. 5. To draw a horizontal and a* V*-parallel in a plane.* (Fig. 191.) As shown by Figs. 173 and 174, any horizontal line has a v. p. parallel to G. L., and—if contained by a plane—must be parallel to the h. t. of the plane, and meet V on its v. t.; therefore any line $b'\,c'$, parallel to G. L., will do for the v. p. of the desired line; the intersection of $P'\,Q$

by $b'\ c'$ will be the v. t. of the line, a vertical through which to G. L. gives c—one point of the plan $c\ b$, whose known direction enables it to be immediately drawn.

A V-parallel is parallel to the v. t. of the plane in which it lies, and meets H on the h. t. of the plane; hence assume d_1^3 as the h. t. of the desired line, project to d' and draw $d'\ a'$ parallel to $P'\ Q$; then $d\ a$, parallel to G. L., represents (with $d'\ a'$) a V-parallel of the plane.

Additional conditions might be assigned to either kind of line, as, for example, the distance from the plane to which the line is parallel; the quadrant; the length of the line, or that it should contain a certain point of the plane.

313. *Prob. 6. Having one projection of a point in a plane, to locate the other projection.* If a point is on a line, its projections are vertically above each other on the projections of the line; if, therefore, the plan, s, (Fig. 191) of the point is given, determine a horizontal line through the point and in the plane. This will be $b\ s\ c$, parallel to $R\ Q$. From c, where it meets G. L., a vertical to $P'\ Q$ gives its v. t., through which draw $c'\ b'$ parallel to G. L. The desired v. p. is then s', on $c'\ b'$ and vertically above s.

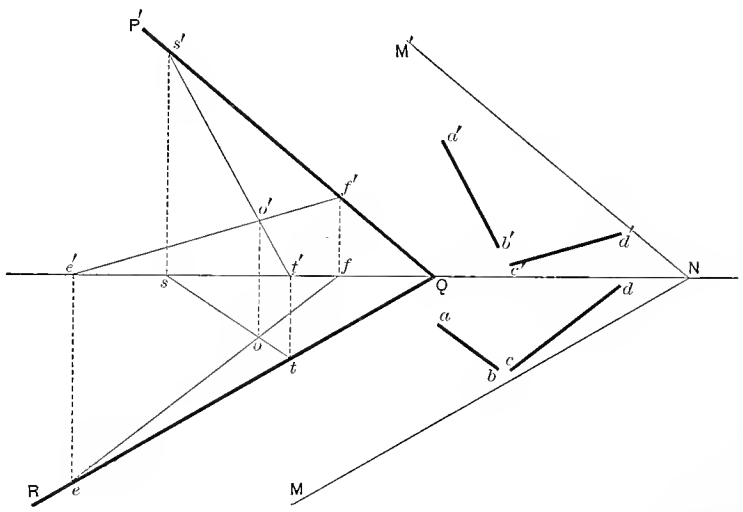
Were the *elevation* of the point given we would find its plan on the h. p. of a V-parallel drawn through the point and in the plane.

314. *Prob. 7. To pass a plane through three points not in the same straight line.* *Any two* of the three lines that would connect the points by pairs would determine the plane by the first case of Prob. 3; while the line joining any two of the points, together with a parallel to it through the third point, becomes the second case of the same problem.

315. *Prob. 8. To pass a plane through one line and parallel to another* draw through any point of the first line a parallel to the second line; such parallel will, with the first line, determine the plane.

316. *Prob. 9. To pass a plane through a given point and parallel to a given plane.* Two lines through the given point and parallel to any pair of lines in the given plane will determine the required plane.

Fig. 192.

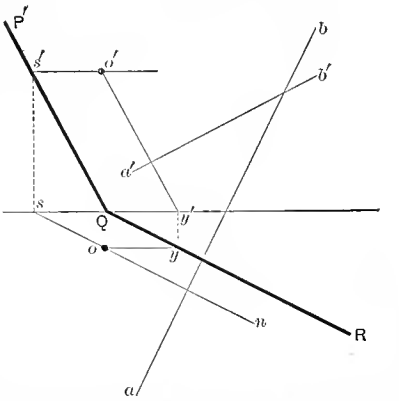


In Fig. 192, with $o\ o'$ as the given point, and $M'\ N\ M$ as the given plane, assume *in the latter* any two lines, $A\ B$ and $C\ D$; parallel to these lines and through $o\ o'$ draw $S\ T$ and $E\ F$, whose traces will determine those of the plane sought.

Since parallel planes have parallel traces, *one* line through $o\,o'$ would suffice. For example, $s\,t$, $s'\,t'$, parallel to $a\,b$, $a'\,b'$, gives the traces t and s', through which draw $R\,Q$ and $Q\,P'$, parallel respectively to the like traces of the given plane.

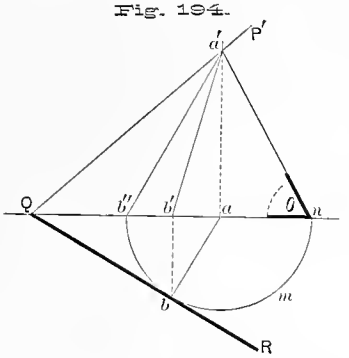
317. *Prob. 10. To pass a plane through a given point and perpendicular to a given line.* Since, by Art. 303, the traces of the desired plane will be perpendicular to like projections of the line, draw through the given point, $o\,o'$, Fig. 193, either a *horizontal* or a V-*parallel* of the desired plane; the trace of either line, and the known directions of the required traces, suffice to solve the problem. The plan, $o\,n$, of a horizontal, will be perpendicular to $a\,b$—the plan of the given line; through s', the v. t. of the horizontal, we draw $P'\,Q$ perpendicular to $a'\,b'$, for the v. t. of the plane; then $Q\,R$ perpendicular to $a\,b$ for the desired h. t. A V-parallel through $o\,o'$ is obtained by drawing $o'\,y'$ perpendicular to $a'\,b'$, and $o\,y$ parallel to G. L.; through y—the h. t. of the V-parallel—we then draw $R\,Q$ in the known direction, then, from Q, the trace $Q\,P'$ perpendicular to $a'\,b'$.

Fig. 193.



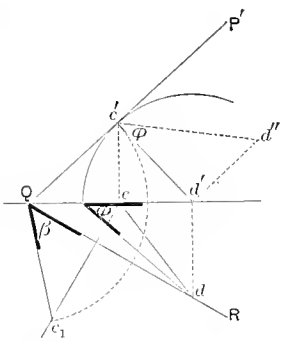
318. *Prob. 11. To determine* (a) *the angles* θ *and* ϕ, *made with* H *and* V *respectively, by a given plane;* (b) *the angle between the traces of the plane.* From the properties of the cone and its tangent plane, mentioned in Art. 307, we may solve the problem by generating a cone with a line of declivity of the plane and ascertaining the inclination of such line.

Fig. 194.



In Fig. 194 let $P'\,Q\,R$ be the plane. The projections $a'\,b'$ and $a\,b$—the latter perpendicular to $R\,Q$—represent a line of declivity of the plane with respect to H. With it, and about $a'\,a$ as an axis, generate a semi-cone. When the generatrix reaches V, either at $b''\,a'$ or $a'\,n$, its inclination to G. L. shows the base angle θ of the cone and therefore the inclination of the given plane.

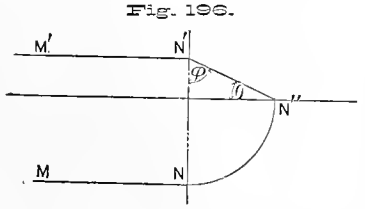
Fig. 195.



With respect to V the construction is analogous. A line of declivity with respect to V has its v. p. perpendicular to $P'\,Q$, (Fig. 195). Using d on the h. t. of the plane, as the vertex of a semi-cone of horizontal axis $d\,d'$, we find the base of the latter tangent to $P'\,Q$ at c'. Carrying c' to the ground line, about d' as a centre, and joining it with d gives the angle ϕ sought. This problem might also be readily solved by rabatting the line of declivity into a plane of projection. Thus, making $d'\,d''$ perpendicular to $c'\,d'$ and equal to $d'\,d$, we find the angle ϕ between $c'\,d'$ and $c'\,d''$.

For a plane parallel to G. L. use an auxiliary *profile* plane, rotating its line of intersection with the given plane as in Fig. 196.

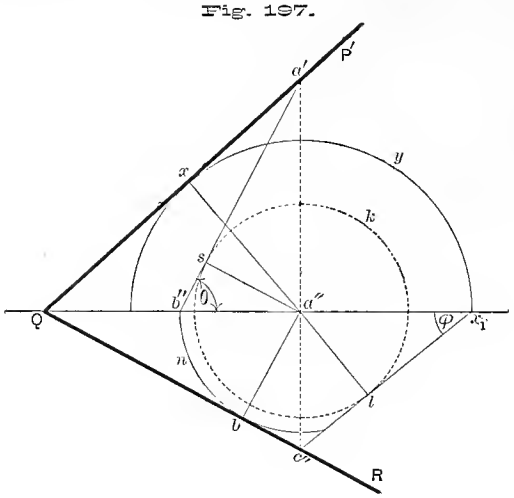
Fig. 196.



(b). *The angle between the traces* is obtained by rabatment of the given plane about either of its traces. In Fig. 195, using trace $R\,Q$ as an axis and rotating $Q\,P'$ about it, any point, c', thus turned, will describe an arc projected in a perpendicular through c to $Q\,R$. Q being on the axis is constant during this rotation, and the distance from it to c' will be the same after as before revolution; therefore cut $c\,c_1$ by an arc, centre Q, radius $Q\,c'$; the desired angle is β or $c_1\,Q\,R$.

319. *Prob. 12. To obtain the traces of a plane, having given its inclinations, θ and ϕ, to* H *and* V *respectively.* This is, obviously, the converse of Prob. 11 and is, practically, the same construction in reverse. The required plane will be tangent, simultaneously, to two semi-cones of base angles θ and ϕ, and having axes (a) in V and H respectively, and (b) in the same *profile plane.*

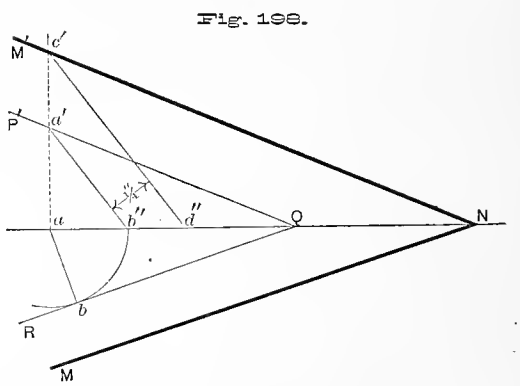
Fig. 197.



Assume in V any vertical line $a'\,a''$ as the axis of the θ-cone, and draw from any point of it, as a', a line $a'\,b''$, at $\theta°$ to G. L; use $a''\,b''$, the plan of this line, as radius of the base of the vertical semi-cone, to which the desired h. t. of the required plane will be tangent. The line $a''\,s$ shows the perpendicular distance from a to the point of intersection of the two elements of tangency of the required plane with the cones; hence the generatrix of the ϕ-cone must, when in H, be at an angle ϕ with G. L., and tangent to arc $s\,k\,l$ of radius $a''\,s$, centre a'', and is therefore $x_1\,c''$. Draw $x_1\,y\,x$ for the half-base of the ϕ-cone. The h. t. of the plane sought is then $R\,Q$, drawn through c'' and tangent to the base of the θ-cone; while the v. t. is $Q\,P'$, tangent to base $x_1\,y\,x$. For the limits of $\theta + \phi$ see Art 301.

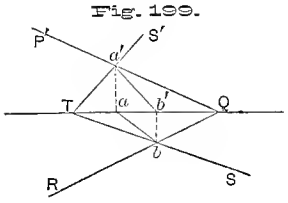
320. *Prob. 13. To draw two parallel planes at a given distance apart.* To draw (Fig. 198) a plane at a distance of $\frac{1}{4}''$ from plane $P'\,Q\,R$, rotate a line of declivity of $P'\,Q\,R$ into V at $a'\,b''$. Draw $c'\,d''$, parallel to and $\frac{1}{4}''$ from $a'\,b''$, to represent (in V) the line of declivity of the plane sought. It meets $a\,a'$ (prolonged) at c', through which draw $c'\,N$ parallel to $P'\,Q$, and from N the trace $M\,N$ parallel to $R\,Q$. $M'\,N\,M$ fulfills the conditions.

Fig. 198.



321. *Prob. 14. To obtain the line of intersection of two planes.* As two points determine a line, we have merely to find two points, each of which lies in both planes, and join them. In Fig. 199 the line *in space* which would join a', the intersection of the vertical traces of the planes, with b, the corresponding point on the horizontal traces, would be the required line. Its projections are $a\,b$, $a'\,b'$.

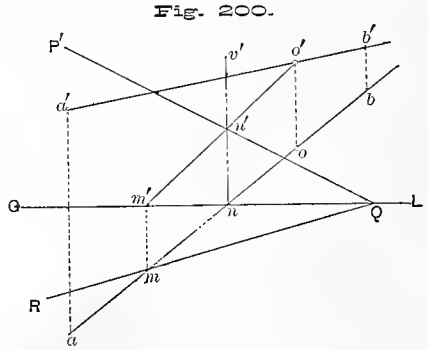
Fig. 199.



Were the H-traces of the planes parallel, the line of intersection would be horizontal. Were the V-traces parallel, their line of intersection would be a V-parallel of each plane.

If both planes were parallel to G. L., a profile plane might advantageously be employed in the solution. The line sought would be parallel to G. L.

Fig. 200.



322. *Prob. 15. To find the point of intersection of a line and plane* find the line of intersection of the given plane by *any* auxiliary plane containing the line; the given line will meet such line of intersection in the desired point. In Fig. 200 $m\,n\,v'$ is an auxiliary vertical plane through the given line $a\,b$, $a'\,b'$. The given plane, $P'\,Q\,R$, intersects the plane

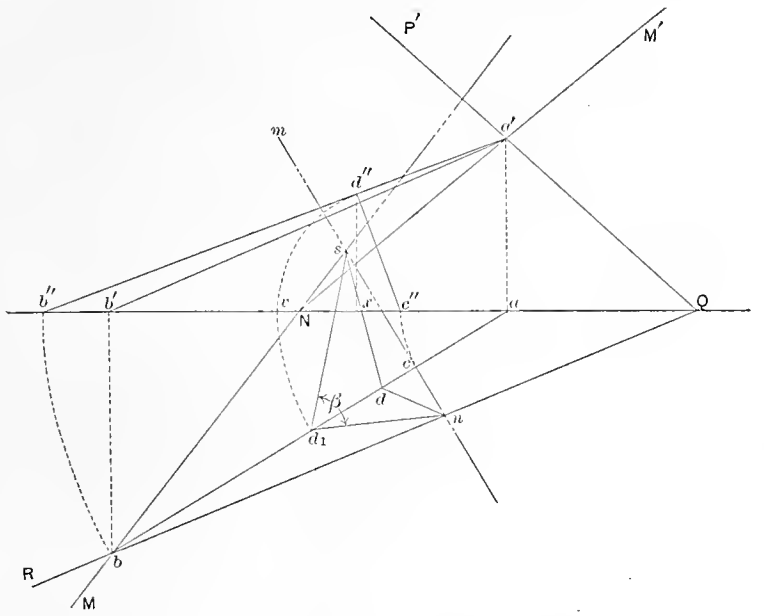
$m\ n\ v'$ in line $m\ n$, $m'\ n'$. The elevations $a'\ b'$ and $m'\ n'$ meet at o'—the v. p. of the point sought, whose h. p. must then be on a vertical through o' and on both the other projections of the lines, hence at o.

323. *Prob. 16. To show the actual angle between two planes.*

A plane perpendicular to the line of intersection of the planes will cut from them lines whose plane angle equals the dihedral angle sought.

Let $a'\ b'$, $a\ b$ be the line of intersection of the planes $P'\ Q\ R$ and $M'\ N\ M$. Any line $m\ n$, drawn perpendicular to $a\ b$, may be taken as the h. t. of an auxiliary plane perpendicular to the line of intersection. The part $s\ n$, included between the H-traces of the given planes, will—with the lines cut from the latter—form a triangle whose vertical angle is that desired. The altitude of this triangle is a line projected on $c\ b$, and—in space—is a perpendicular from c to $A\ B$. To ascertain its length rabat the H-projecting plane of $A\ B$ into V, about $a'\ a$ as an axis. The point c appears at c'', and the line $A\ B$ at $a'\ b''$. The altitude sought appears in actual size at $c''\ d''$, perpendicular to $a'\ b''$. From c lay off on $c\ b$ the length $c\ d_1$ equal to $c''\ d''$; $s\ d_1\ n$, or β, is the required angle.

Fig. 201.



A vertical from d'' to G. L., thence an arc to $c\ b$ (from centre a), gives d, the plan of the vertex of the angle in space.

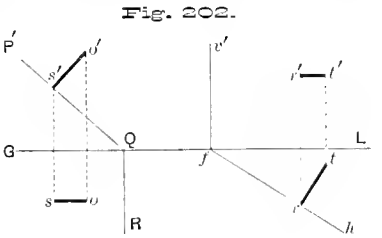
When both planes are parallel to G. L. use an auxiliary profile plane in solving.

324. *Prob. 17. To determine the angle between a line and plane* let fall a perpendicular from any point of the line to the plane; the angle between the perpendicular and the given line will be the complement of the desired angle. The principles involved are those of Arts. 303 and 311.

325. *Prob. 18. To show in its true length the distance from a point to a plane.*

If the plane is perpendicular to H *or* V the distance is evident without any construction. Thus, in Fig. 202, a perpendicular through the point to the plane is seen in its true length, $o'\ s'$, because in space it is parallel to V.

Fig. 202.



Similarly $t\ r$ is the actual length of the perpendicular from $t\ t'$ to the vertical plane $v'\ f\ h$.

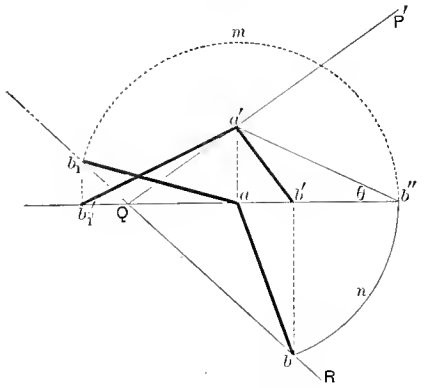
When the plane is oblique to both H *and* V let fall a perpendicular from the point to the plane; find by Prob. 15 the intersection of such perpendicular with the plane, and show, by Prob. 1 (b), the real length of the distance between the original point and the one thus obtained.

326. *Prob. 19. To draw a line of given inclination in a plane.* We have seen, in Art. 301, that we can assign to the line no greater inclination than that of the plane. Starting with the line *in* V (Fig. 203), make its intersection, a', with $P'\ Q$, the vertex of a cone whose base angle θ is the inclination assigned to the line. With $a\ b''$ as a radius describe the base of this cone. This cuts

$R\ Q$—the h. t. of the plane—in b and b_1, each of which, joined in space to a', would give a line fulfilling the conditions.

The projections of the two solutions are $a'\ b'$, $a\ b$; $a'\ b'_1$, $a\ b_1$.

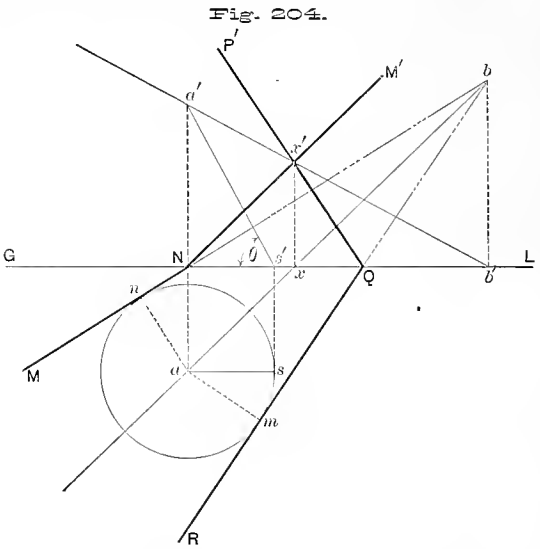
Fig. 203.



If the line is to pass through a given point of the plane make that point the vertex of the cone employed in the construction, taking the initial position of the generatrix parallel to V.

327. *Prob. 20. To obtain the traces of a plane containing a given line and making a given angle with* H *or* V.

Fig. 204.



Make any point on the line the vertex of a cone whose base angle is the assigned inclination; the required plane will be tangent to such cone, and its traces will pass through the like traces of the given line.

In Fig. 204 let $a\ b$, $a'\ b'$ be the given line. On it take any point $a\ a'$ as the vertex of a cone generated by a line $a'\ s'$, $a\ s$, inclined $\theta°$ to H and, at first, parallel to V. Since the H-traces of the two possible planes fulfilling the conditions would be tangent to the like trace of the the cone, draw from b—the h. t. of the given line—the tangents $b\ R$ and $b\ M$ to the circle $m\ s\ n$. These cut G. L. at Q and N, each of which is then joined with x'—the v. t. of the given line—to obtain the v. t. of one of the desired planes.

328. *Prob. 21. To determine the projections of two intersecting lines, having given their inclinations to* H *or* V, *and their angle with each other.*

Fig. 205.

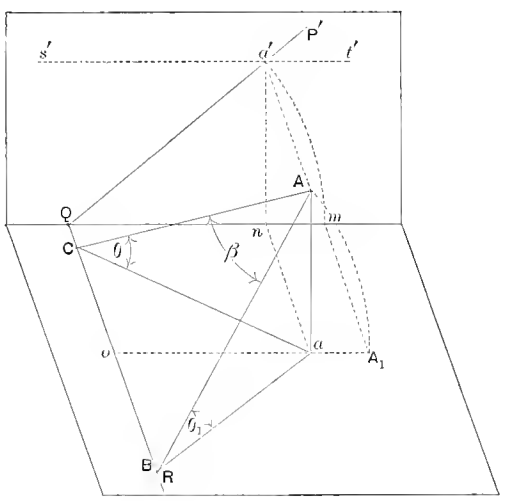
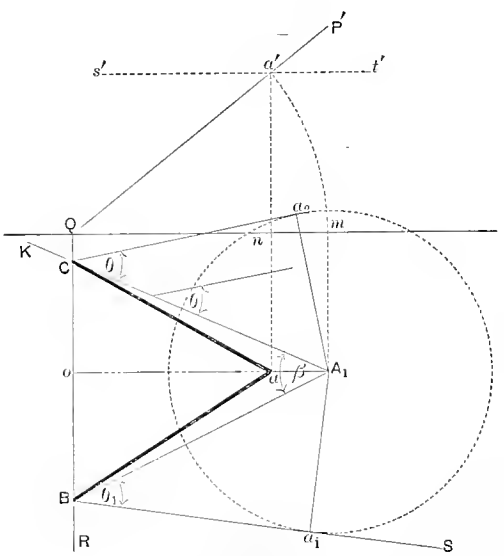


Fig. 206.



It will be seen from Fig. 205 that, in space, either line $A\ B$, $A\ C$, will be the hypothenuse of a right-angled triangle whose plane is perpendicular to that plane of projection relative to which the

inclinations are given; that the altitudes (perpendiculars) of these triangles will not only be equal but coincide (as in $A\,a$), and that the angles θ and θ_1, at their bases, will be the given inclinations to the plane of projection.

Let the two lines, $A\,C$ and $A\,B$, be inclined $\theta°$ and $\theta_1°$ respectively to H, and also make with each other some angle β; to find their projections. From any point on H, as A_1 (Fig. 206), draw two lines, A_1B and A_1K, making the angle β with each other. At *any* point on A_1B, as B, draw a line $B\,S$, making with $A_1\,B$ an angle equal to the inclination to H of the line $A\,B$ in space. From A_1 draw $A_1\,a_1$ perpendicular to $B\,S$. With $A_1\,a_1$ as a radius, and A_1 as a centre, describe a circle. Tangent to this circle draw a line meeting $A_1\,K$ at C, at an angle equal to the other given inclination, $\theta°$. B and C are then the H-traces of the lines $A\,B$ and $A\,C$; and the line $B\,C$ will form part of the h. t. of their plane. For convenience take the vertical plane of projection perpendicular to the plane of the lines, so that the triangle $A\,B\,C$ will be projected upon it in a straight line. This condition will be met by taking G. L. perpendicular to $B\,C$. On V draw $s'\,t'$ parallel to G. L., and at a distance from it equal to $A_1\,a_1$ or $A_1\,a_2$, either of which represents the actual height of the point A in space, when the lines $A\,B$ and $A\,C$ make the given angles with H. With Q as a centre and radius $Q\,m$ (m is the v. p. of A_1) describe an arc, $m\,a'$, intersecting $s'\,t'$ at a'. This arc is the v. p. of the arc that A_1 describes about $R\,Q$ in reaching its space-position. $A_1\,o$ is the h. t. of the plane in which A_1 rotates. The h. p. of A is then at a—the intersection of $o\,A_1$ by $a'\,n\,a$, the projector-plane of A. The desired plans of the given lines are $a\,B$ and $a\,C$, while $a'\,Q$ is their common elevation.

Limits of the conditions that may be assigned. The sum of the given inclinations and the angle between the lines will reach its *maximum*, 180°, when the lines lie in a profile plane.

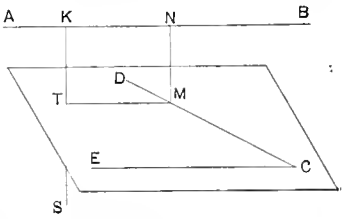
The length of but one of the lines may be predetermined. If assigned, such length must be made the hypothenuse of the first triangle constructed. The longer line will, obviously, make the smaller angle.

329. *Prob. 22. To determine the traces of two mutually perpendicular planes, having given their inclinations to* H *or* V. Two planes will be mutually perpendicular if either contains a line that is perpendicular to the other. To solve this problem we therefore determine, by the first step of Prob. 12, the traces of a plane at one of the given inclinations, θ, to H; by Art. 303 erect a perpendicular to this plane; by Art. 327, obtain the traces of a plane containing such perpendicular and tangent to a cone (a) whose vertex is on the perpendicular and (b) whose base angle is the other assigned inclination.

The planes will intersect in a *horizontal* line when the sum of their dihedral angle and their inclinations to H has its minimum value, 180°.

330. *Prob. 23. To determine the common perpendicular to two lines not lying in the same plane.* Only a pictorial representation of the various steps is shown (Fig. 207) their orthographic counterparts having been already fully described in detail. Let $A\,B$ and $C\,D$ be the given lines. Pass a plane through either line parallel to the other. The parallelogram represents such a plane, determined by $C\,D$ and $C\,E$, the latter a parallel to $A\,B$ and drawn through any point of $D\,C$. By Art. 303, drop a perpendicular $K\,T\,S$, from *any* point of $A\,B$ to the auxiliary plane, and obtain its trace T on the latter by Art. 322. $T\,M$, drawn parallel to $A\,B$, will be the projection of the latter on the plane. $M\,N$, a parallel to $T\,K$ through the intersection of $T\,M$ and $D\,C$, will be the desired perpendicular, whose true length may be ascertained as in Art. 308.

Fig. 207.



DEFINITIONS AND VARIOUS CLASSIFICATIONS OF THE MORE IMPORTANT LINES AND SURFACES.

In our graphical constructions the draughtsman is concerned—not with the *contents* of solids—but with the *form* and *relative position* of the *surfaces* that bound them, the *lines* in which such surfaces meet, and the *points* which are the intersections or extremities of such lines. After the acquaintance with the relations between figures and their projections which may now be assumed if the preceding articles of this chapter have been mastered, a sort of bird's-eye glimpse into the province of their application is in order; the remainder of this chapter is therefore devoted to definition, classification, illustration, and some of the important properties of the principal lines and surfaces with which the mathematician and draughtsman have to deal, one object being to enlarge the student's horizon as to the extent of the mathematico-graphical field and to give him some idea of its attractiveness.

ALGEBRAIC AND TRANSCENDENTAL LINES AND SURFACES.—DEGREE.—ORDER.—CLASS.

331. *Algebraic.—Transcendental.* Taking up first the distinctions of Analytical Geometry (for definition see page 4) we find that a line or surface is *algebraic* if the relations of the co-ordinates of its points can be expressed by an equation which may be reduced to a finite number of terms, involving positive, integral powers of the variables. If, however, its equation involves other functions—as trigonometrical, circular, exponential, logarithmic—the curve or surface is called *transcendental.*

332. *Degree.—Order.* The *degree* of an algebraic curve or surface is that of the equation expressing it. The term *order* is used synonymously with *degree.*

The degree, or order, of an algebraic curve is the same as the maximum number of points in which it can be cut by a plane, and for a plane curve is the number of times it can be met by a right line in its plane.

The straight line is the only line of the first order.

The conic sections are the only mathematical curves of the second order.

Of the other curves treated in Chapter V the Witch of Agnisi and the Cissoid are of the *third* order; the Limaçon, Cardioid, Trisectrix, Cartesian ovals and Cassian ovals are of the *fourth;* and the remainder—Helix, Sinusoid, the Trochoids in general, Catenary, Tractrix, Involute and the Spirals—are *transcendental.*

Since transcendental equations expand into an infinite series their degree is assumed infinite, from which we conclude that a straight line may have an infinite number of intersections with a transcendental plane curve; analogously for a plane and a transcendental space curve. The former may be illustrated by the cycloid, (Fig. 100), which is not completed by rolling the generating circle out *once* on a straight line, but consists, theoretically, of an infinite series of similar arcs.

333. When all the plane sections of a surface are lines of a certain order the surface is of the same order.

A *plane* is the only surface of the *first* order.

Conicoids or *quadrics* (see Art. 367) are *all* and *the only* surfaces of the *second* order.

Of the other surfaces defined in the following articles the Conoid of Plücker (Cylindroid of Cayley) is of the *third* order; the Cyclide of Dupin, Cono-cuneus of Wallis, the Cylindroid of Frézier, Corne de Vache, Conchoidal Hyperboloid of Catalan and the Torus are of the *fourth;* and the others are *transcendental.*

334. *Class.* The *class* of a *plane curve* is the number of tangents that can be drawn to it from a point in its plane.

The *class* of a *non-plane curve* is the number of osculating planes* that can be passed to it through any point in space.

The *class* of an algebraic *surface* equals the number of possible tangent planes to it through any line in space.

LINES AND SURFACES AS ENVELOPES.—GENERATION BY CONTINUOUS MOTION.—REVOLUTION.—TRANSPOSITION.

335. *Envelopes.* If the points of a line are obtained as the consecutive intersections of the curves of a series, the line would be tangent to all the curves of the system, and would be called their *envelope.* On page 22 we find the parabola as the envelope of a series in which the right line appears as the variable curve. Similarly we find the parallel curve to the lemniscate (page 66) as the envelope of all circles of the same radius, whose centres are on that curve.

Analogously, a surface may be defined as the envelope of the various positions taken by another surface. Thus the right cone is the envelope of all the possible planes containing a given point on a line and making a constant angle with the line.

336. *Lines and surfaces regarded as the result of continuous motion.* By the conception with which the draughtsman is mainly concerned, a line is considered as generated by the motion of a *point;* a *surface* by the motion of a *line.* He regards a straight line as generated by a point moving always in the same direction†; a curve as the path of a point whose direction of motion changes continually; a plane as the surface generated by a moving straight line which glides along a fixed straight line while passing always through a fixed point.

337. A *curve* is *plane* if any four consecutive positions of its generating point lie in one plane.

338. A *space curve* or *curve of double curvature* or *non-plane curve* is any curve not plane; that is, a curve generated by a moving point, no four consecutive positions of which lie in the same plane.

Chief among space curves is the *ordinary helix,* whose construction is given in Art. 120. After it may be mentioned the *conical helix* (see Art. 191) and the *spherical epicycloid,* the latter being the theoretical outline of the teeth of bevel gears, and generated by the motion of a point on an element of a cone which rolls on a cone having the same slant height, their vertices being common.

The spherical epicycloid evidently lies on the surface of a sphere whose radius equals an element of either cone.

339. Whether it be straight or curved, the moving *line* that generates a *surface* is called the *generatrix,* and in *any* of its positions is an *element* of the surface. Its motion may be either of *revolution* or *transposition.*

340. *Revolution* implies a straight line called an *axis,* which may or may not be in the plane of the revolving line. Each point of the moving line describes a circle whose plane is perpendicular to the axis and whose centre is on the axis.

Any plane *containing* the axis of a surface of revolution is called a *meridian plane* and cuts the surface in a *meridian curve.*

All planes *perpendicular* to the axis of a surface of revolution intersect it in circles called *parallels,* the smallest of which, so long as of finite radius, is called the *circle of the gorge.*

Any surface of revolution may also be generated as a surface of transposition.

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* An *osculating* plane to a non-plane curve contains three consecutive points of the curve.

† Frequently also defined as *the shortest distance between two points.* If we take into account the generalized notions of modern geometry as to *direction* and *distance* we might define a *straight line* as *the line which is completely determined by two points;* a *curve* as *any line not straight:* a *plane* as *the surface containing all straight lines connecting its points by pairs; parallel straight lines* as *non-intersecting straight lines in a plane.*

341. *Transposition* may be defined as any motion other than revolution, and involves certain guiding *lines* called *directrices*, along which the generatrix glides; and, frequently, certain *surfaces* called *directors*, with respect to whose elements the generatrix of the new surface takes definite positions.

The surface-director most frequently occurring is the *cone*, including its special case, the *plane*. Every element of a cone-director will have a parallel element on the other surface.

RULED SURFACES.

342. Whether the motion be of revolution or transposition, any surface that can have a *right line generatrix* is called a *ruled surface*. In contradistinction, all others are called *double curved surfaces*, since *any* plane section thereof must be a curve.

As to *curvature*, ruled surfaces other than the *plane* are called *single curved*.

343. The motion of a right line involves the consideration of "conditions of restraint" and "degrees of freedom." If it is to move so as always to intersect another line, the generatrix is under *one* condition of restraint, but, as to motion, has still two degrees of freedom. Require it in all its positions to intersect *two* lines and it has still one degree of freedom. Impose a third condition, either to move so as always to intersect each of three lines—either straight or curved—or, while intersecting two lines, to be parallel to some element of a surface-director, or to be tangent to a given surface, and we have then determined the line in position *thus far*, that we have located it upon a certain surface. Impose one more—a fourth condition—and the limit of *motion* is reached, the line becoming a particular line on that particular surface.

344. *Developable Surfaces.* A surface which can be directly rolled out or unfolded upon a plane, without undergoing distortion, is called a *developable surface.*

345. *Plane-sided figures* evidently belong to this division and, after the *pyramid* and *prism*, the more important are the *regular polyhedrons*, or solids bounded by *equal*, *regular polygons* whose planes are all inclined to each other at the same angle. Of these, *five* are convex to outer space, i. e., no face produced will cut into the solid. They are the *tetrahedron*, *octahedron* and *icosahedron*, bounded, respectively, by *four*, *eight* and *twenty* equal, equilateral *triangles;* the *cube* or *hexahedron*, whose *six* faces are equal *squares;* and the *dodecahedron*, whose surface consists of *twelve* equal, regular, *pentagons.*

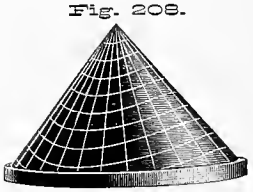
By allowing re-entrant angles the number of regular solids has been increased by *four*, named after their investigator, Poinsot, and frequently also called *star polyhedra*, from their form.*

346. A *single curved surface* is called a *developable* or *torse* if *consecutive positions* of its rectilinear generatrix lie *in the same plane.*

If *not only consecutive* elements but *any pair* lie in the same plane we have either a *cone* or a *cylinder.*

When *only consecutive* elements lie *in the same plane* we have developable surfaces of great variety, but whose common characteristic is, that they are generated by a straight line moving so that in every position it is a tangent to a curve of double curvature.

Fig. 208.



ELLIPTICAL CONE.

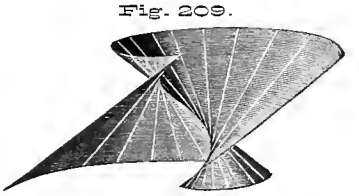
That such a surface is developable may be thus demonstrated without reference to any figure. Suppose a, b, c and d to be four *consecutive* points of a curve of double curvature. The line joining a and b is—by the ordinary definition—a tangent to the curve. Similarly $b\,c$ and $c\,d$ are tangent and consecutive. Tangent $a\,b$ meets tangent $b\,c$ at b and their plane is tangent at b to the generated surface, since it contains

* For their construction, as also of the convex regular solids, see Rouché et De Comberousse, *Traité de Géométrie*, Part II, Book VII.

two lines, each of which is tangent to the surface at the same point. Also $b\,c$ intersects $c\,d$ at c. But, since four consecutive points of a curve of double curvature cannot lie in the same plane, the tangent $a\,b$ does not meet tangent $c\,d$. The plane $a\,b\,c$ may, however, be rotated on $b\,c$ into coincidence with plane $b\,c\,d$, and this process repeated indefinitely, bringing the entire surface into one plane without disturbing the relative position of consecutive elements.

In Fig. 209 we have the *developable helicoid*, a representative surface of the family just described. Its elements (the white lines) are all tangent to a helix. In Arts. 186 and 187 its upper and lower outlines are described.

Fig. 209.



Like the cone, complete surfaces of this family consist of two *nappes* or sheets. In the cone both nappes meet at the vertex. On other developables they meet on the curve whose tangents constitute the surface.

Could the nappes of such a surface be as readily separated as those of a cone either of them could as directly be developed upon a plane by rolling contact.

The moving line generates both nappes simultaneously, its point of tangency separating it into the portions on upper and lower nappes respectively.

Whatever the mathematical name for the curve of double curvature employed it has the following names as a line of the developable surface: (a) *cuspidal edge*, since any plane section of the surface will have a *cusp* or point, on that curve; (b) *edge of regression*, since along it the surface is most contracted.

347. The family of surfaces just described, as, in fact, *all* developable surfaces *but two*, must be *surfaces of transposition;* for a straight line can have *but three* positions with respect to an axis about which it is to *revolve*:—

(a) it may be *parallel* to the axis, when a *cylinder of revolution* will result, *developable*, since consecutive elements lie in the same plane, because parallel;

(b) it may *intersect* the axis, giving the *cone of revolution*, *developable*, because consecutive elements lie in the same plane by virtue of intersecting;

(c) it may make an angle with the axis without intersecting it, giving consecutive positions not lying in the same plane.

348. *Warped Surfaces or Scrolls.* A *warped surface* or *scroll* is a ruled surface in which *consecutive elements do not lie in the same plane.*

All warped surfaces *but one* must be surfaces of transposition: for we have seen, from the last article, that any straight line that generates a surface of *revolution* either *is* or *is not* in the same plane with the axis; and that if the former is the case a developable surface is generated: if, then, a *warped surface of revolution* be possible, it must be due to the revolution of a right line about an axis not in its plane—evidently the only other possible alternative. The proof that such motion fulfills the essential condition of a warped surface is as follows:—

(a) Consecutive positions of the revolving line *do not intersect*, as—having no point *on* the axis—each *point* must describe a circle about the axis and cannot, therefore, coincide with its preceding position;

(b) Consecutive positions *are not parallel*, for if parallel in space they would be in projection; while reference to Fig. 77, which is a plan and elevation of the surface in question, shows that they are projected as *consecutive tangents* to the projection of the circle of the gorge ($A\,B\,C\,D$), rendering parallelism impossible.

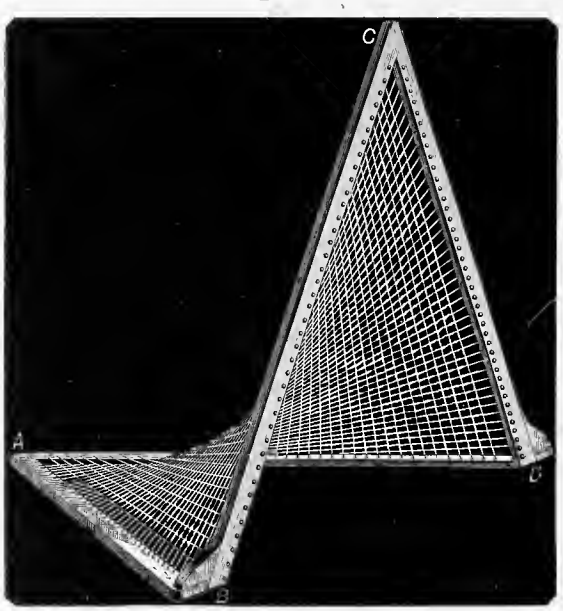
Neither intersecting nor being parallel, consective elements do not, therefore, lie in the same plane.

The warped surface of revolution is also called the *hyperboloid of revolution* since it may be generated by the revolution of its meridian curve—an hyperbola—about its conjugate (or imaginary) axis. If the asymptote to the hyperbola be simultaneously revolved with it, a surface called an asymptotic cone will be generated. The curves cut by a plane from the hyperboloid and its asymptotic cone will be of the same mathematical nature.

349. *Warped Surfaces of Transposition.*. The most important of these are the *hyperbolic paraboloid;* the *elliptical hyperboloid* or *hyperboloid of one sheet; conoidal surfaces* such as the *cono-cuneus of Wallis* and the *conoid of Plücker;* the *warped helicoid;* the *conchoidal hyperboloid of Catalan;* the *cylindroid of Frézier*, and the *warped arch*, also called the *corne de vache.*

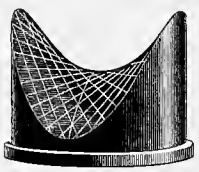
Referring to Art. 343, regarding the number of conditions that may be imposed on a moving

Fig. 210.



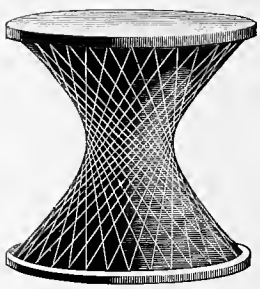
HYPERBOLIC PARABOLOID.

Fig. 211.



HYPERBOLIC PARABOLOID.

Fig. 212.



ELLIPTICAL HYPERBOLOID.

line, let us first consider all the possible surfaces having *three straight directrices.* Evidently such directrices, no two of which lie in one plane, either *are* or *are not* parallel to some plane; for *any two* of them will invariably be parallel to some plane, and the third either *is* or *is not* parallel to that plane. Should the former be the case the resulting surface would be the *hyperbolic paraboloid,* which may also be defined as the warped surface having *two straight directrices and a plane director;* if, however, the directrices have the second position supposed, the surface generated is called the *hyperboloid of one sheet* or the *elliptical hyperboloid:* these two (and their special forms) evidently exhaust the possibilities as to surfaces with *only straight directrices.*

350. The hyperboloid and hyperbolic paraboloid are further interesting as being the only surfaces which may be *doubly-ruled*, that is, may have two different sets of rectilinear elements; for in such a surface it must be possible for *either directrix* of one set of elements to become the generatrix of a second set; hence, to become in its turn an *element*, each generatrix must be *straight*, thus restricting the possibilities—as to number of surfaces—to the two named.

351. In a doubly-ruled warped surface *each* element of one generation, while intersecting no other elements of its own set, meets *all* of the other set; and *any three* of either set might be taken as directrices for the other.

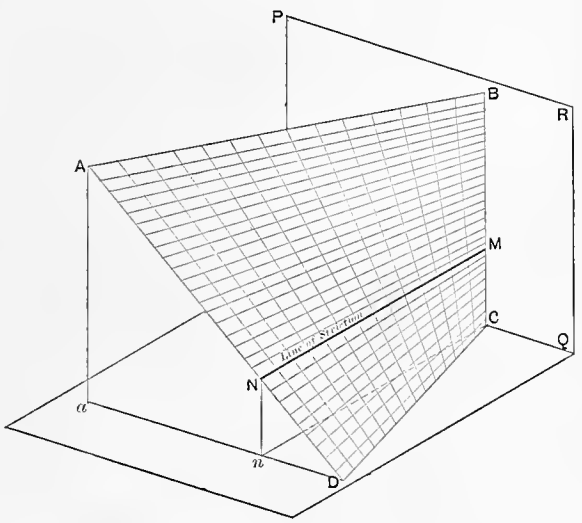
352. Each set of elements of the hyperbolic paraboloid is—by the second definition of the surface—parallel to its own plane director. Section-planes parallel to the line of intersection of the plane directors will cut the surface in *parabolas;* while other plane sections are *hyperbolas.*

The curvature of the hyperbolic paraboloid is called *anticlastic.* Its typical, saddle-like form is best illustrated by Fig. 211.

353. A *line of striction* is the line—straight or curved—containing the shortest distance between consecutive elements of a warped surface. For the hyperbolic paraboloid it is that element of one set which is perpendicular to the plane director of the other set of elements. (See Fig. 213.)

354. *A conoidal surface* is generated by a line moving parallel to a plane director and so as always to intersect a fixed right line or axis, fulfilling, at the same time, an additional condition, either of meeting some fixed curve or of being tangent to a fixed surface.

Fig. 213.



HYPERBOLIC PARABOLOID WITH A LINE OF STRICTION.

Fig. 214.

RIGHT CONOID.

THE CONO-CUNEUS OF WALLIS.

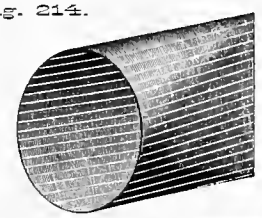
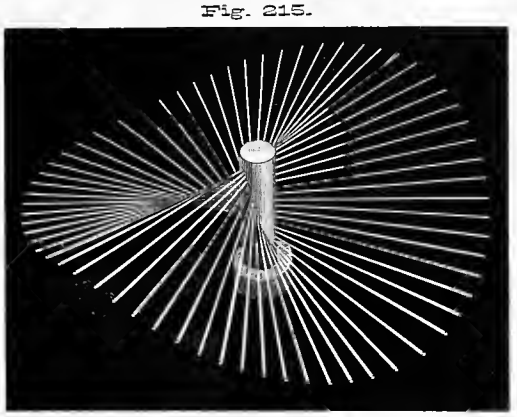


Fig. 215.



CONOID OF PLÜCKER.

When the axis or straight directrix is perpendicular to the plane director we have a *right conoid,* and the axis becomes the *line of striction.*

355. With a *circle* as the curved directrix; an axis *equal* to, *parallel to* and *directly opposite* a diameter of the circle; and with a plane director perpendicular to the axis, we would obtain the right conoid known as the *Cono-cuneus of Wallis.* This surface (Fig. 214) has been employed with pleasing effect in architectural constructions, in towers, arches and for the faces of wing-walls.

Plane sections parallel to the curved directrix are ellipses whose longer axes are equal to the straight directrix.

356. Another interesting surface of the same family is the *Conoid of Plücker**, whose mechanical and kinematic properties are treated by Prof. R. S. Ball in his *Theory of Screws;* also by Clifford in his *Elements of Dynamic.* The name *cylindroid,* under which it appears in those works, was suggested by Cayley, but had much earlier been pre-empted by Frézier for the surface defined in Art. 360.

* See Plücker's *Neue Geometrie des Raumes,* p. 97; also Mannheim's *Géométrie Descriptive,* p. 435.

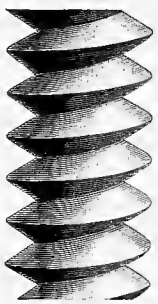
Fig. 215 represents a model of the conoid under consideration; but for the exact mathematical surface "the diameter of the central cylinder must be conceived to be evanescent and the radiating wires must be extended to infinity." (*Ball*).

Were $A\,B$ and $C\,D$ (Fig. 213) to be the common directrices of hyperbolic paraboloids whose plane directors were various positions of $P\,R\,Q$, rotated about the common perpendicular $B\,C$, their lines of striction would be elements of a conoid of Plücker.

The same surface will result if a right line be moved so that, while perpendicular to and intersecting the axis of a circular cylinder, it shall follow a double-curved directrix obtained by wrapping around the cylinder a *sinusoid* (Art. 171) or *harmonic curve*, two waves of which reach once around.

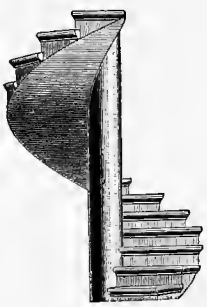
357. The *warped helicoid* has for its directrices a helix and its axis, the generatrix making a constant angle with the latter. The last condition is equivalent to saying that the surface has a cone director.

Fig. 216.



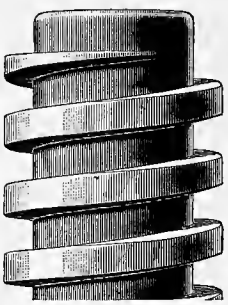
OBLIQUE HELICOIDS.

Fig. 217.



RIGHT HELICOID.

Fig. 218.



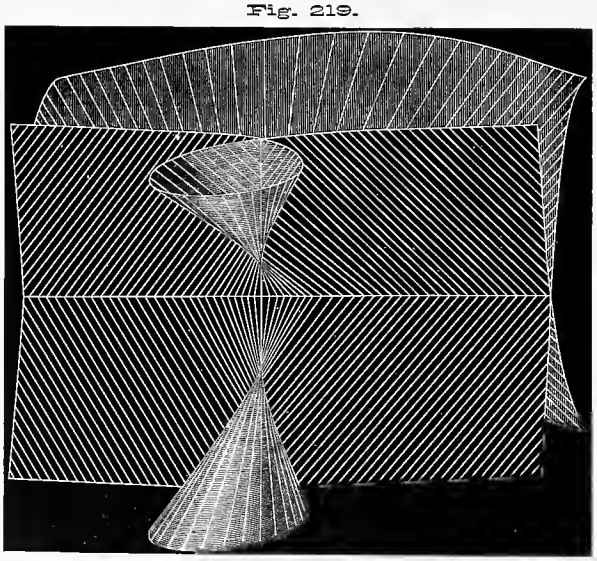
RIGHT HELICOIDS.

If the angle made by the elements with the axis is *acute*, the surface is called the *oblique helicoid.* It is the acting surface of ordinary triangular-threaded screws, and of many screw propellers.

Plane sections perpendicular to the axis of this surface are Archimedean spirals.

358. When the elements are perpendicular to the axis the cone director becomes a plane director and the surface is called the *right* helicoid, familiar to all as the under surface of a spiral staircase. It is the acting surface of a square-threaded screw and, frequently, of screw-propellers.

Fig. 219.



CONCHOIDAL HYPERBOLOID OF CATALAN.

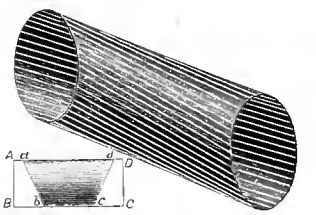
The right helicoid also belongs among conoidal surfaces, and its axis is—like that of the conocuneus—a line of striction.

359. *The conchoidal hyperboloid*, invented by Catalan, has two non-intersecting, rectilinear directrices—one horizontal, the other vertical—the generatrix making a constant angle with the latter.

Planes parallel to both directrices will cut hyperbolas from the surface, while horizontal sections will be conchoids. (See Arts. 193 and 196).

360. *The cylindroid of Frézier* has a plane director and two curved directrices. In its usual form it may be imagined to be thus derived from a cylinder of revolution: Suppose a cylinder *A B C D* (Fig. 220) on H and parallel to V—the plane director; for curved directrices employ the ellipses cut from the cylinder by non-parallel, vertical, section-planes, *a b*, *c d*, taken on opposite sides of some vertical right section; if *one* of these ellipses be shifted *vertically*, *in its own plane*, the lines joining the new positions of its points with their former points of connection on the other directrix will be elements of a cylindroid. This surface has been suggested for the soffit (under surface) of a descending arch.

Fig. 220.



CYLINDROID OF FRÉZIER.

Any plane *containing* the line in which the planes of the curved directrices intersect will cut congruent[1] curves from the cylinder and cylindroid; while planes *parallel to* such line will cut plane sections of the same area.

361. *The warped arch* or *corne de vache* has three linear directrices, one straight the others curved; the latter are equal circles in parallel planes, while the straight directrix is perpendicular to the planes of the circles and passes through the middle point of the line joining their centres. In oblique or skew arch construction one of the best known methods is that in which the soffit of the arch is a Corne de Vache.[2]

Fig. 221.

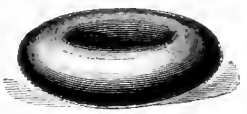
CORNE DE VACHE.

DOUBLE CURVED SURFACES.

362. *Double Curved Surfaces* are surfaces that cannot be generated by a right line.

363. *Double Curved Surfaces of Revolution.* The *sphere* is the most familiar example under this head, the generatrix being a semi-circle and the axis its diameter. After it come the *ellipsoids*—the *prolate spheroid* and the *oblate spheroid*—generated by rotating an ellipse about its major or minor axis respectively; the *paraboloid of revolution*, generatrix—a parabola, axis—that of the curve; the *hyperboloid of revolution of separate nappes*, formed by rotating the two branches of an hyperbola about their transverse axis; and the *torus*—annular or not—generated by revolving a circle about an axis in its plane but not a diameter. (See Fig. 222; also Arts. 112–114).

Fig. 222.



THE TORUS.

364. The revolution of other plane curves—as the involute, tractrix, cycloid, conchoid, etc., gives double curved surfaces of frequent use in architectural constructions and the arts.[3]

365. *Double Curved Surfaces of Transposition.* Of the innumerable surfaces possible under this head we need only mention here the *serpentine*, generated by a sphere whose centre travels along a helix; the *ellipsoid of three unequal axes*, which would result from turning an ellipse about one of its axes in such manner that, while remaining an ellipse, its other axis should so vary in length that its extremities would trace a second ellipse; the *elliptical paraboloid*, whose plane sections perpendicular

[1] Congruent figures, if superposed, will coincide throughout.

[2] For a comparison of the relative merits of these methods refer to *Skew Arches*, by E. W. Hyde. (Van Nostrand's Science Series, No. 15.)

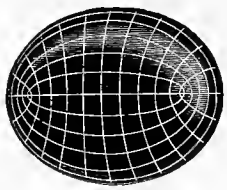
[3] See Note, p. 64; also Art. 203.

to the axis are ellipses, while sections containing the axis are parabolas; the *elliptical hyperboloid of one nappe*, generated by turning a variable hyperbola about its real axis so that its arc shall follow an elliptical directrix: the *elliptical hyperboloid of two nappes*, analogous to the two-napped hyperboloid

Fig. 223.

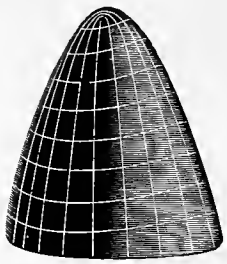
SERPENTINE.

Fig. 224.



ELLIPSOID.

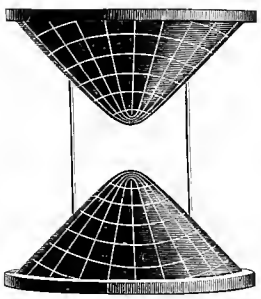
Fig. 225.



ELLIPTICAL HYPERBOLOID.

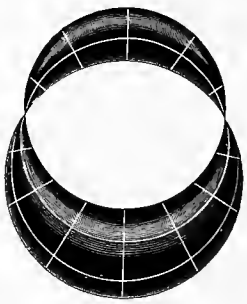
of revolution, but having elliptical instead of circular sections perpendicular to its axis; and the *cyclide**, whose lines of curvature (see Art. 378) are all circles and each of whose normals intersects two conics—an ellipse and hyperbola—whose planes are mutually perpendicular and having the foci of each at the extremities of the transverse axis of the other. The cyclide is the envelope of all

Fig. 226.



ELLIPTICAL HYPERBOLOID OF TWO NAPPES.

Fig. 227.



THE CYCLIDE OF DUPIN.

spheres (a) having their centres on one of the conics and (b) tangent to any sphere whose centre is on the other conic.

The torus is a special case of the cyclide.

TUBULAR SURFACES.—QUADRIC SURFACES OR CONICOIDS.

366. Among the surfaces we have described, the *torus* and *serpentine* belong to the family called *tubular*, since each is *the envelope of a sphere of constant radius.*

367. Surfaces whose plane sections are invariably conics are called *conicoids* or *quadrics.* These are the cone, cylinder and sphere; ellipsoids; hyperboloids of one and two nappes; elliptic and hyperbolic paraboloids. In theory, *all* conicoids are *ruled* surfaces; but on some the right lines are imaginary, while they are real on the cone, cylinder, hyperboloid of one sheet and hyperbolic paraboloid.

TANGENTS AND NORMALS TO CURVES.

368. A *tangent* to a curve is a right line joining two *consecutive* points of the curve.

A *normal* to a curve is the right line perpendicular to the tangent at the point of tangency.

Both tangent and normal to a *plane curve* lie in its plane.

* For mathematical, optical and other properties of the cyclide see Salmon, *Geometry of Three Dimensions*, and the writings of J. Clerk Maxwell.

If the curve is the intersection of a surface by a plane, the tangent to it at any point will be the line of intersection of the plane of the curve by a plane tangent to the surface at the given point.

369. The tangent at any point of a *non-plane* curve, when the latter is the intersection of two surfaces, is the line of intersection of two planes, each of which is tangent—at the given point—to one of the surfaces.

LINES AND PLANES, TANGENT AND NORMAL TO SURFACES.

370. A straight line joining two consecutive points of a surface is a *tangent* to it.

371. *A tangent plane* to a surface at any point is the locus of all the right lines tangent to the surface at that point.

372. A right line perpendicular to a tangent plane at the point of tangency is a *normal* to the surface.

373. Any plane containing the normal cuts from the surface a *normal section.*

TANGENT PLANES TO RULED SURFACES.

374. Since any element of a ruled surface fulfills the condition (Art. 310) necessary to make it a tangent to the surface, it may be taken as one of the two right lines necessary to determine a plane, tangent to the surface at any point of the element.

In case the surface is *doubly ruled* the two elements through the point would determine the tangent plane.

375. In general, the line which—with an element—would determine a tangent plane to a ruled surface at a given point, would be a tangent, at that point, to *any curve* passing through the latter and lying on that surface.

376. *A plane, tangent to a developable surface* at any point, would, therefore, be determined by the element containing the point and, preferably, by a tangent to the base at the extremity of the element, since to such a surface a tangent plane has *line* and not merely *point* contact.

There is, however, this difference to be noted between the two kinds of developable surfaces; a tangent plane to a cylinder or cone is tangent the entire length of an element, while, in the case of a surface of which the developable helicoid is representative, a plane tangent to the lower nappe along an element becomes, in general, a *secant* plane to the upper nappe, and vice versa.

377. *The tangent plane at any point of a warped surface* is found by Arts. 374 and 375, and is in general a secant plane as well, tangency being along an entire element only in special cases.

TANGENT PLANES TO DOUBLE CURVED SURFACES.

378. *A tangent plane to a double curved surface* has, usually*, but one point of contact with it.

If a normal to the surface can readily be drawn then the tangent plane may be determined most simply on the principle that it will be perpendicular to the normal, at the given point. This method is especially applicable in problems of tangency to a *sphere*, since the radius to the point of tangency is the normal to the surface; also to any *tubular surface*, since such may be regarded as generated by the motion of a sphere, and at any point of the circle of contact of sphere and tubular surface they would have a common tangent plane.

* For an exception refer to Fig. 74, page 34, where a *bi-tangent* plane to an annular torus is seen in profile view as the line *Q P.*

In general, by taking the the *two* simplest curves that could be drawn through the point of desired tangency and upon the surface, the tangent plane would be determined by the tangents to these curves at that point. In Chapter V methods are given for drawing tangents to the more important mathematical curves, among which we would find nearly all of the plane sections of the surfaces defined in this chapter.

INTERSECTION OF SURFACES.

379. The line of interpenetration of two intersecting surfaces may be found by cutting them by a series of auxiliary surfaces; the two sections of the former, cut by any one of the latter, will meet (if at all) in points of the desired line of intersection.

The surfaces should be so located with respect to H and V as to facilitate the drawing of the auxiliary sections; and the latter should, if possible, be straight lines or circles, in preference to other forms less easy to represent.

RADII AND LINES OF CURVATURE. — OSCULATORY CIRCLE AND PLANE. — GEODESICS.

380. *Radius of Curvature.* A circle is *osculatory* to a non-circular curve when it has three consecutive points in common with it. Its radius is called the *radius of curvature* of the curve, for the middle one of the three points.

381. *Line of Curvature.* It is ascertained by analysis that among all the possible normal sections at any point of a surface *two* may be found, mutually perpendicular, whose radii of curvature are respectively the maximum and minimum for that point; such sections are called *principal sections*, and their radii the *principal radii* for that point.

If upon any surface a line be so drawn that the tangent to it at any point lies in the direction of one of the principal sections at that point, the line is called a *line of curvature* of the surface.

Since at every point of a surface two principal sections are possible, there may also be drawn through each point *two* lines of curvature, intersecting each other at right angles. Such curves are shown in white lines on nearly all the preceding figures illustrating mathematical surfaces.

382. *Geodesics.* When a circle is osculatory to a non-plane curve its plane is called an *osculating plane.* At every point of a *geodesic line* on a surface the osculating plane is normal to the surface.

Either the greatest or shortest distance between two points of a surface would be measured on the geodesic passing through them. Since the maximum or minimum distance between two points on a *sphere* would be measured on the *great circle* containing them, such circle would be the geodesic on that surface.

A helix is the geodesic on a cylinder.

CHAPTER X.

CONVENTIONAL METHODS OF GROUPING PROJECTIONS.—WORKING DRAWINGS.

383. In the preceding chapter the principal surfaces which occur in engineering and architectural construction, and which are capable of mathematical definition, have been illustrated and described, and a scientific foundation laid for the graphical solution of problems relating thereto.

In dealing with some of these surfaces, by way of illustration of general principles, the figures will, for the most part, be found to be *fully dimensioned*, placing them under the head of what have already been described, viz., *working drawings*. While it is important and customary that working drawings should be strictly "to scale" throughout, yet the workman is, ordinarily, held responsible only for following the dimensions given in figures by the draughtsman.

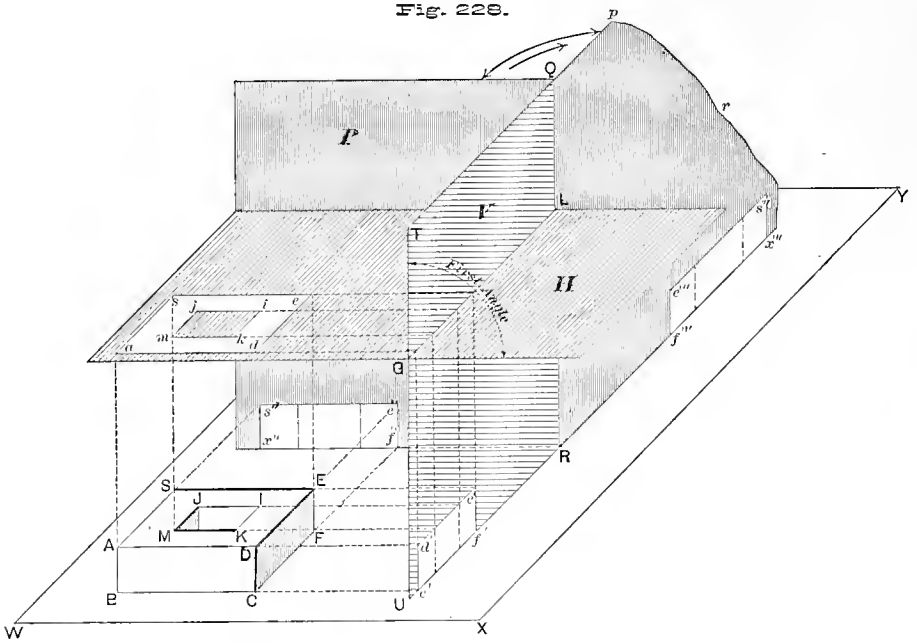
Although in sheet-metal work, particularly in blast-furnace construction, very difficult graphical problems often occur, there are probably none that surpass in complication those occasionally arising in designing masonry arches. To solve such the draughtsman should not only be familiar with all that the preceding chapter contains, but have a thorough course in stereotomy, especially as applied to stone-cutting. His practical problem in any case would be (a) to draw the *projections* of the arch or other design and derive from them (b) *templets* showing the exact form of the *edges* of the stone; (c) *bevels* showing the *dihedral angles* between *faces*; (d) *patterns*, in sheet-metal, showing the exact *form* of the various *faces* of the stone. The work of the mason is greatly facilitated if, in addition to the foregoing, he is furnished with a pictorial representation of the stone upon which he is to work, drawn either in isometric or oblique projection. While the stone-cutter is not expected to understand the theory of projections the machinist must be familiar with it to a certain extent; but, in order to simplify for him the "reading" or comprehension of working drawings, a self-interpreting method of grouping the views has been adopted by a large number of leading American mechanical engineers, although the old method has still ardent advocates among prominent designers and will probably remain the favorite for stereotomy and for purely mathematical work. We shall call the new method that of the *Third Angle*, the old—or *First Angle*—method being that with which the student has become familiar in the last chapter, and with which this chapter concludes.

384. *Third Angle Method.* This is illustrated by the planes and the hollow, prismatic block $A\ B\ C\ D-F$, in Fig. 228. Projectors from its points to the three planes of projection give $d'\ e'\ f'\ c'$ for the *elevation*, $a\ s\ e\ d$ for the *plan* and $e''f''x''s''$ for the *profile view* or *side elevation*. Rotation of the profile plane about $Q\ R$ as an axis, and in the direction indicated by the arrow, carries the side elevation into V at $e'''f'''x'''s'''$. Rotation of the first angle part of H *downward*, and the rear of H *upward*, on axis G. L., would carry the plan, $a\ d\ e\ s$, into V *above* $d'\ e'\ f'\ c'$.

The lettering of each projection makes it represent that side of the object which is *nearest* to the plane on which the projection is made. This is one of the distinctive features of the method under consideration, the other being that the rotations supposed bring the view of the *top* of the object *above* the front elevation; the view of the *right-hand side* of the object (the observer is supposed to be looking in the direction $X\ W$) appears at the *right* of said elevation.

Were it desirable to project the face $A\ B\ C\ D$ this method would involve a vertical plane on $W\ X$, upon which the view would appear; then revolution anti-clockwise would bring the projection into V at the *left* of $d'\ e'\ f'\ c'$.

Fig. 228.



We might obtain similar relative positions of the projections, were the object *in front* of V and *below* the plane $Q\ T\ X\ N$ (Fig. 229), by elevating that plane sufficiently to rotate it downward about $Q\ T$ so as to bring the shaded plan $a\ d\ e\ s$ into V above the elevation.

Fig. 229.

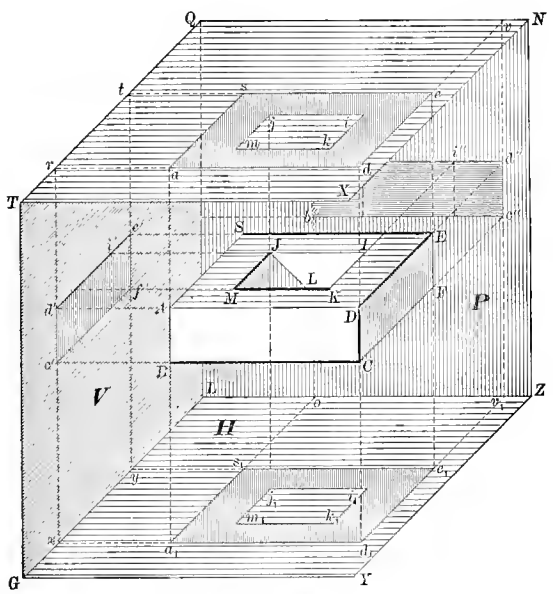
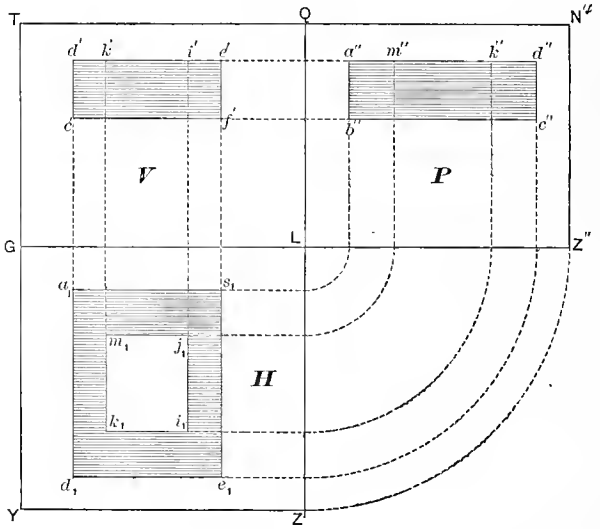


Fig. 230.

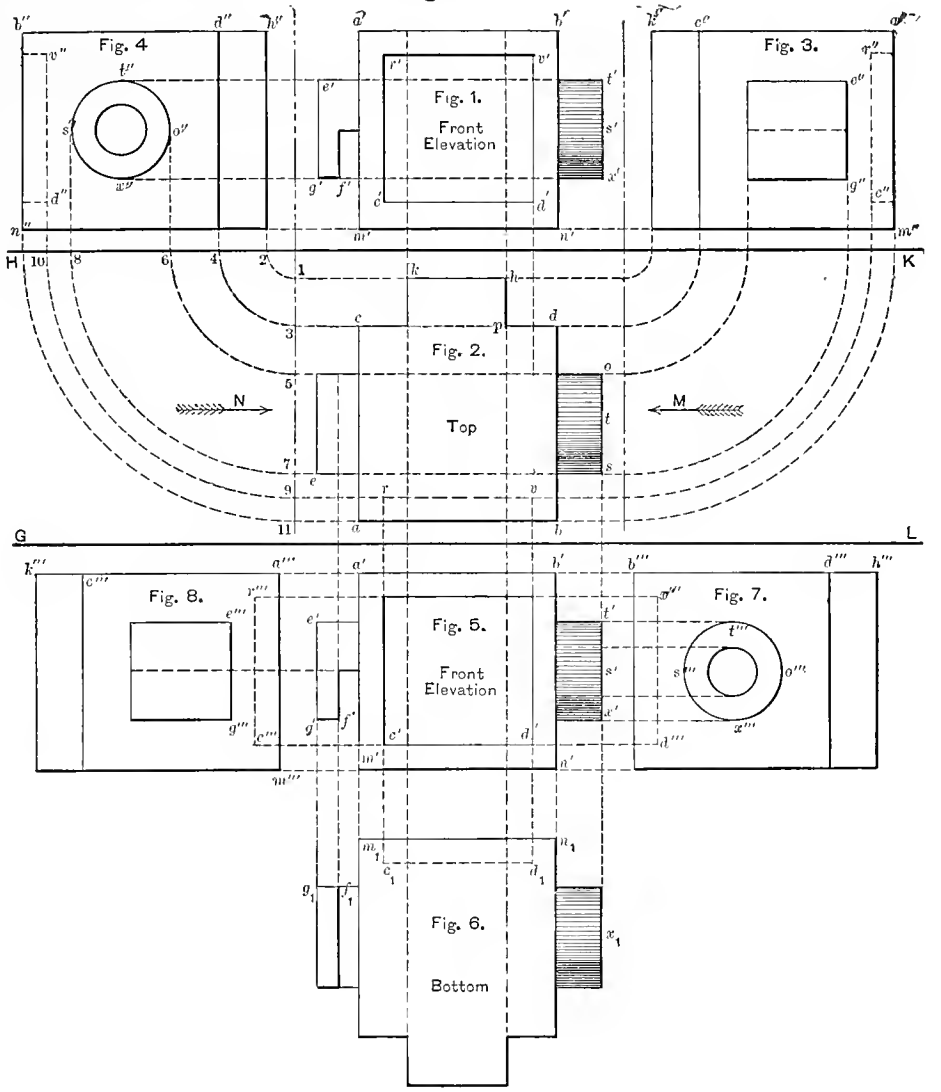


385. *First Angle Method.* Disregarding $Q\ T\ X\ N$ (Fig. 229) we have the object and planes illustrating the first angle method, the lettering of each projection showing that it represents the side of the object *farthest* from the plane.

In the ordinary representation the same object would be shown by its three views as in Fig. 230. In the front and side elevations the short-dash lines indicate the invisible edges of the opening in the block.

386. For the sake of more readily contrasting the two methods a group of views is shown in Fig. 231, all above G. L. illustrating an object by the First Angle system, while all below *H K* represents the same object by the Third Angle method.

Fig. 231.



When looking at Figures 1, 2, 3 and 4 the observer queries: What is the object, *in space*, whose *front* view is like Fig. 1, *top* view is Fig. 2, *left* side is like Fig. 3 and *right* side like Fig. 4? For the view of the left side he might imagine himself as having been at first between *G* and *H*, looking in the direction of arrow *N*, after which both himself and the object were turned, together, to the right through a ninety-degree arc, when the same side would be presented to his view in Fig. 3. Similarly, looking in the direction of the arrow *M*, an equal rotation to the left, as indicated by the arcs 1-2, 3-4, 5-6, etc., would give in Fig. 4 the view obtained from direction

M. His mental queries would then be answered about as follows: Evidently a cubical block with a rectangular recess—$r'v'd'c'$—in front; on the rear a prismatic projection, of thickness $p\,h$ and whose height equals that of the cube; a short cylindrical ring projecting from the right face of the cube; an angular projecting piece on the left face.

In Fig. 2 the line $r\,v$ is in short dashes as, in that view, the back plane of the recess $r'v'.d'c'$ would be invisible. In Fig. 4 the back plane of the same recess is given the letters, $v''d''$, of the edge nearest the observer from direction *M.*

Third Angle Method. Ignoring all above line $H\,K$ we have in Fig. 5 the same front elevation as before, but *above it* the view of the *top; below it* the view of the *bottom*, exactly as it would appear were the object held before one as in Fig. 5, then given a ninety-degree turn, around $a'\,b'$, until the under side became the front elevation.

Fig. 7 may as readily be imagined to be obtained by a shifting of the object as by the rotation of a plane of projection; for by translating the object to the right, from its position in Fig. 5, then rotating it to the left 90° about $b'\,n'$, its right side would appear as shown. The group 2-5-6-7-8 might, therefore, be conceived as derived by entirely surrounding the object by a system of mutually perpendicular planes, parallel to its main lines; a projection made—on each plane—of the side nearest it on the object; and the planes then rotated into V in such manner, as to right- or left-handed rotation, that the view presented to us is that which we would have of the object itself, were we looking squarely at the face projected.

387. We shall for the present employ the Third Angle Method, preceding its application by a general resumé of terms, abbreviations and hints as to notation and execution of problems.

H, V, P	the *horizontal*, *vertical* and *profile* planes of projection respectively.
H-projector	that projecting line which gives the *horizontal projection* of a point.
V-projector	the projecting line giving the projection of a point *on V.*
Projector-plane	the profile plane containing the projectors of a point.
h. p.	the *horizontal projection* or *plan* of a point or figure.
v. p.	the *vertical projection* or *elevation* of a point or figure.
h. t.	*horizontal trace*, the intersection of a line or surface with H.
v. t.	*vertical trace*, the intersection of a line or surface with V.
H-traces, V-traces	plural of horizontal and vertical traces respectively.
G. L.	*ground line*, the line of intersection of V and H.
V-parallel	a line parallel to V and lying in a given plane.
A horizontal	any horizontal line lying in a given plane.
Line of declivity	the steepest line, with respect to one plane, that can be drawn in another plane.
Rabatment	revolution into H or V about an axis in such plane.
Counter-rabatment or revolution	restoration to original position.

388. *For problems relating solely to the point, line and plane, and not involving solids*, the *given lines* should be fine, continuous, black; *required lines* heavy, continuous, black or red; *construction lines* fine, continuous red or short-dash black; *traces of an auxiliary plane*, or *invisible traces* of *any* plane, in dash-and-three-dot lines, red or black.

For Problems relating to Solid Objects.

(1) *Pencilling.* Exact; generally completed for the whole drawing before any inking is done*; the work usually from centre lines, and from the larger—and nearer—parts of the object to the smaller or more remote.

(2) *Inking of the Object.* Curves to be drawn before their tangents; fine lines uniform and drawn before the shade lines; shade lines next and with one setting of the pen, to ensure uniformity.

Regarding *tapering shade lines* see Art. 111. The tapering should immediately follow the drawing of the curve.

(3) *Location of Shade Lines. In architectural work* these would be drawn in accordance with a given direction of light. (See chapter on Shadows.)

*See Art. 66 for exception; also Arts. 25, 65, 73 and 115 for more detailed information on many of the other points.

In *American machine-shop practice* the *right-hand* and *lower* edges of a *plane* surface are shade lines if they separate it from *invisible* surfaces. Indicate *curvature* by *line-shading* if not otherwise sufficiently evident. (See Art. 115.)

(4) *Invisible lines of the object*, black, invariably, in dashes about one-tenth of an inch in length. – – – – – – – – –

(5) *Inking of lines other than of the object.* When *no colors* are to be employed the following distinctions as to kind of line are those most frequently made. The lines may preferably be drawn in the order mentioned.

Centre lines, an alternation of dash and two dots —— · · —— · · —— · · —— · · —— · · ——

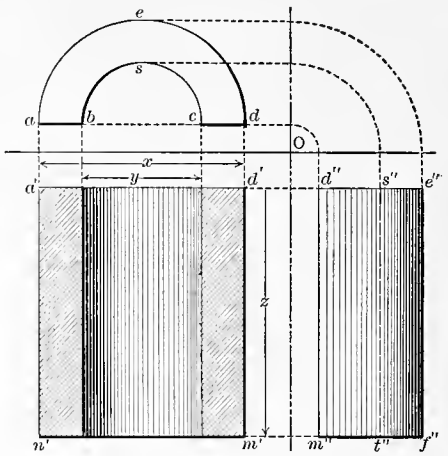
Dimension lines, a dash and dot alternately, with opening left for the dimension —— · —— · —— —— · ——

Ground line, (when it cannot be advantageously omitted) a continuous heavy line ————————

Construction and other explanatory lines, in short dashes – – – – – – – –

390. *Working drawing of a vertical, semi-cylindrical pipe:* outer diameter, x; inner diameter, y; height, z.

Fig. 234.



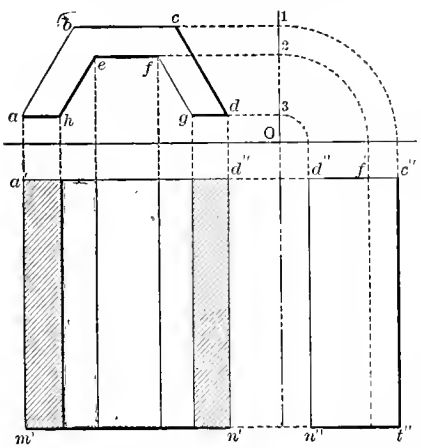
For the *plan* draw concentric semi-circles $a\ e\ d$ and $b\ s\ c$, of diameters x and y respectively, joining their extremities by straight lines $a\ b$, $c\ d$. At a distance apart of z inches draw the upper and lower limits of the elevations and project to these levels from the points of the plan.

In the side view the thickness of the shell of the cylinder is shown by the distance between $e''\ f''$ and $s''\ t''$—the latter so drawn as to indicate an invisible limit or line of the object.

Ordinarily the line shading would be omitted, the shade lines on the various views sufficing to convey a clear idea of the form.

391. *Half of a hollow, hexagonal prism.* In a semi-circle of diameter $a\ d$ step off the radius three times as a chord, giving the vertices of the plan $a\ b\ c\ d$ of the outer surface. Parallel to $b\ c$, and at a distance from it equal to the assigned thickness of the prism, draw $e\ f$, terminating it on lines (not shown) drawn through b and c at 60° to $a\ d$. From e and f draw $e\ h$ and $f\ g$, parallel respectively to $a\ b$ and $c\ d$. Drawing $a'\ c''$ and $m'\ t''$, as upper and lower limits, project to them as in preceding problems for the front and side elevations.

Fig. 235.

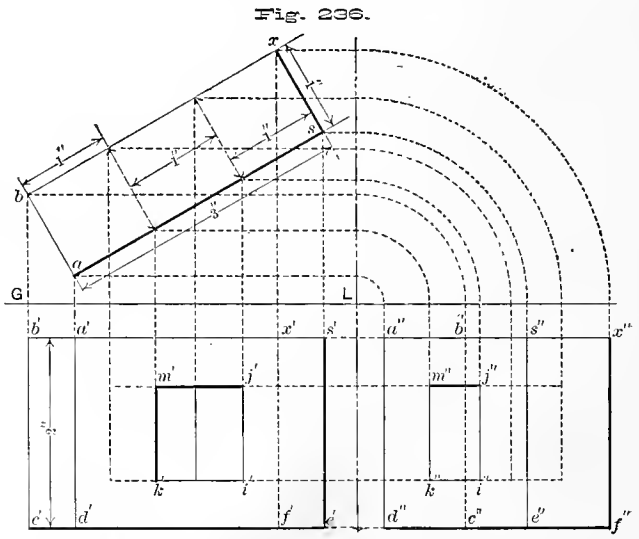


392. *Working drawing of a hollow, prismatic block, standing obliquely to the vertical and profile planes.*

Let the block be $2'' \times 3'' \times 1''$ outside, with a square opening $1'' \times 1'' \times 1''$ through it in the direction of its thickness. Assuming that it has been required that the two-inch edges should be vertical, we first draw, in Fig. 236, the plan $a\ s\ x\ b$, $3'' \times 1''$, on a scale of 1 : 2. The inch-wide opening through the centre is indicated by the short-dash lines.

For the elevations the upper and lower limits are drawn 2″ apart, and a, b, s, x, etc., projected to them. The elevations of the opening are between levels $m'\ m''$ and $k'\ k''$, one inch apart, and equidistant from the upper and lower outlines of the views. The dotted construction lines and the lettering will enable the student to recognize the three views of any point without difficulty.

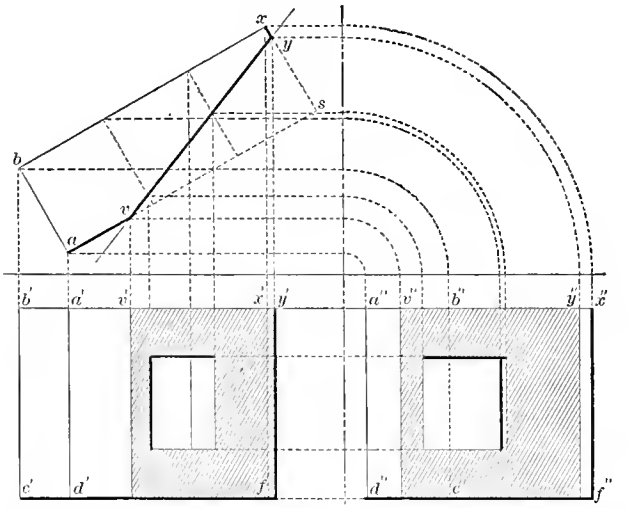
Fig. 236.



393. In Fig. 237 we have the same object as in Fig. 236, but represented as cut by a vertical plane whose h. t. is $v\ y$. The parts of the block that are actually cut by the plane are shown in section-lines in the elevations. This is done here and in some later examples merely to aid the student in understanding the views; but *in engineering practice section-lining is rarely done on views not perpendicular to the section plane.*

394. *Suppression of the Ground Line.* In machine drawings it is customary to omit the ground line, since the *forms of* the various views—which alone concern us—are independent of the distance of the object from an imaginary horizontal or vertical plane. We have only to remember that all elevations of a point are at the same level; and that if a ground line or trace of any vertical plane is wanted, it will be perpendicular to the line joining the plan of a point with its projection on such vertical plane.

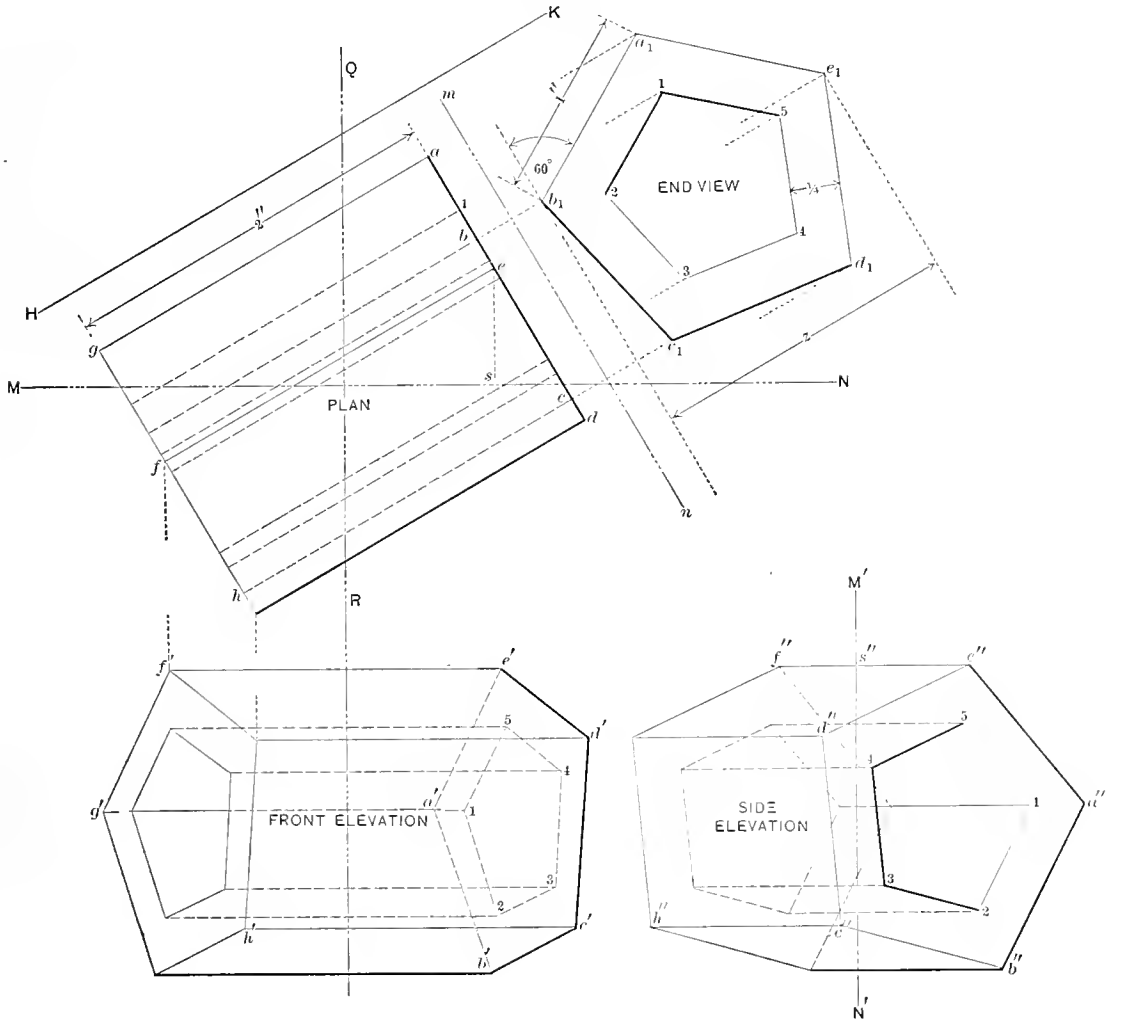
Fig. 237.



395. *To draw a hollow, pentagonal prism,* 2″ long; *edges* to be horizontal and inclined 35° to V; *base,* a regular pentagon of 1″ sides; one *face* of the prism to be inclined 60° to H.

In Fig. 238 let $H\,K$ be parallel to the plans of the axis and edges; it will make 35° with a horizontal line. Perpendicular to $H\,K$ draw $m\,n$ as the h. t. of an auxiliary vertical plane upon

Fig. 238.



which we suppose the base of the prism projected. In end view all the faces of the prism would be seen as lines, and we may draw $a_1 b_1$, one inch long and at 60° to $m n$, to represent the face whose inclination is assigned. Completing the inner and outer pentagons, allowing, for example, $\frac{1}{4}''$ for the distance between faces, we have the end view complete. The plan is then obtained by drawing parallels to $H K$ through all the vertices of the end view and terminating all by vertical planes, $a d$ and $g h$, parallel to $m n$ and 2″ apart.

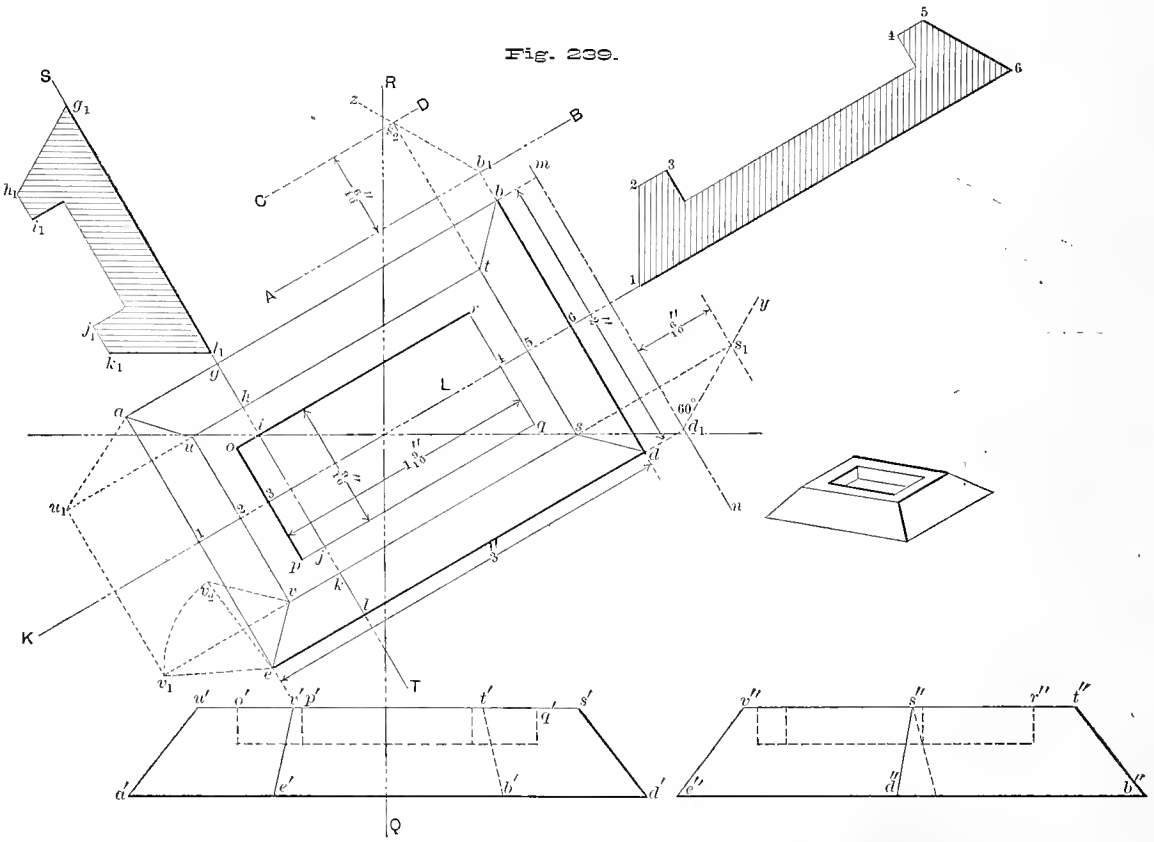
The elevations will be included between horizontal lines whose distance apart is the extreme height z of the end view; and all points of the *front elevation* are on verticals through their plans and at heights derived from the end view. These heights are estimated from the level of b_1.

Since the profile plane is omitted, we take $M' N'$ to represent the trace upon it of the auxiliary, central, vertical plane whose h. t. is $M N$. All points of the *side elevation* are then at the same level as in the front elevation, and at distances to the right or left of $M' N'$ equal to the perpendicular distances of their *plans* from $M N$. For example, $e'' s''$ equals $e s$.

The shade lines are located on the end view on the assumption that the observer is looking towards it in the direction of $H K$.

396. *To draw a truncated pyramidal block having a rectangular recess in its top;* angle of sides, 60°; lower base a rectangle 3″ × 2″, having its longer sides at 30° to the horizontal; total height $\frac{6}{10}''$; recess $1\frac{9}{10}'' \times \frac{9}{10}''$, and $\frac{1}{4}''$ deep. The small oblique projection on the right of Fig. 239 shows, pictorially, the figure to be drawn.

Fig. 239.



The plan of the lower base will be the rectangle $a b d e$, 3″ × 2″, whose longer edges are inclined 30° to the horizontal.

Take $A\,B$ and $m\,n$ as the H-traces of auxiliary vertical planes perpendicular to the side and end faces of the block. Then the sloping face whose lower edge is $d\,e$, and which is inclined 60° to H, will have $d_1\,y$ for its trace on plane $m\,n$. A parallel to $m\,n$ and $\frac{6}{10}''$ from it will give s_1, the auxiliary projection of the upper edge of the face $s\,v\,e\,d$, whence $s\,v$—at first indefinite in *length*—is derived, parallel to $d\,e$. Similarly the end face $b\,t\,s\,d$ is obtained by projecting $d\,b$ upon $A\,B$ at b_1, drawing $b_1\,z$ at 60° to $A\,B$ and terminating it at s_2 by $C\,D$, drawn at the same height ($\frac{6}{10}''$) as before. A parallel to $b\,d$ through s_2 intersects $v\,s_1$ at s, giving one corner of the plan of the upper base, from which the rectangle $s\,t\,u\,v$ is completed with sides parallel to those of the lower base.

As the recess has vertical sides, we may draw its plan $o\,p\,q\,r$ directly from the given dimensions, and show the depth by short-dash lines in each of the elevations.

The ordinary elevations are derived from the plan as in the preceding problems; that is, for the front elevation, $a'\,u'\,s'\,d'$, by verticals through the plans; for the side elevation, $e''\,v''\,t''\,b''$, the distances to the right or left of s'' equal those of the plans of the same points from $s\,i$, regarding the latter as the h. t. of a central plane parallel to V.

The plane $S\,T$ of *right section, perpendicular to the axis* $K\,L$, cuts the block in a section whose true size is shown in the tinted figure $g_1\,h_1\,k_1\,l_1$, and whose construction hardly needs detailed treatment after what has preceded. The shaded longitudinal section, on central vertical plane $K\,L$, also interprets itself by means of the lettering.

The true size of any face, as $a\,u\,v\,e$, may be shown by rabatment about a horizontal edge, as $a\,e$. As v is actually $\frac{6}{10}''$ above the level of e, we see that $v\,e$ (in space) is the hypothenuse of a triangle of base $v\,e$ and altitude $\frac{6}{10}''$. Construct such a triangle, $v\,v_2\,e$, and with its hypothenuse $v_2\,e$ as a radius, and e as a centre, obtain v_1 on a line through v and perpendicular to $a\,e$ and representing the path of rotation. Finding u_1 similarly, we have $a\,u_1\,v_1\,e$ as the actual size of the face in question.

If more views were needed than are shown the student ought to have no difficulty in their construction, as no new principles would be involved.

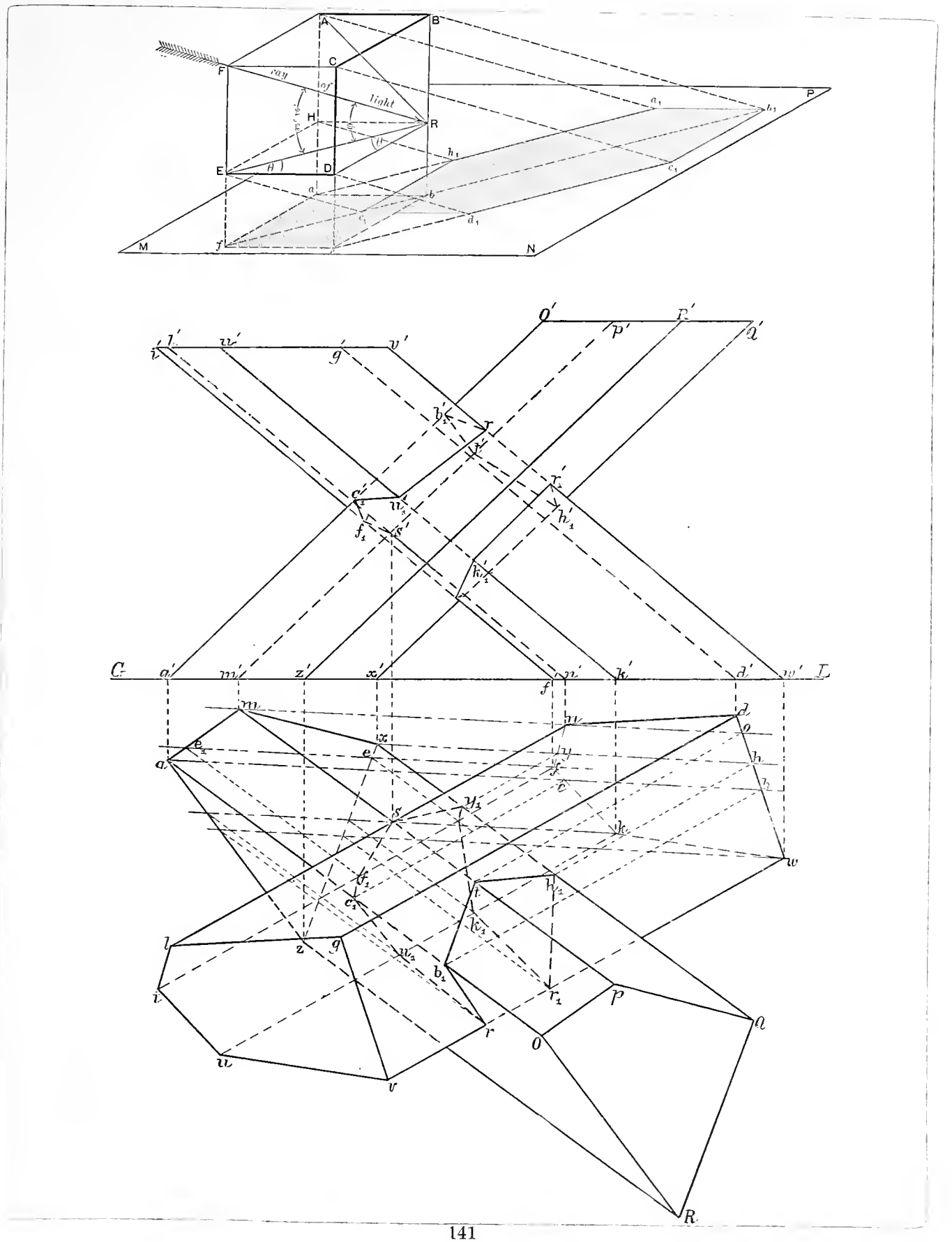
397. The last two problems illustrate all of the features of the Third Angle method, as practically applied in machine drawing and design. They may be thus summarized:

(a) The various views of the object are grouped around the front elevation, that of the top being *above* it, that of the bottom *below* it, and analogously for the projections of the right and left sides.

(b) The ground line is omitted, although its direction may be readily determined by inspection.

(c) Central reference-planes are taken through the various views, and, in each view, the distance of any point from the trace of the central plane of that view is obtained by direct transfer, with the dividers, of the distance between the same point and reference-plane, as seen in some other view — usually the plan.

THE remaining pages contain some illustrations for which the text is not complete, but regarding which oral instruction will be given; also a large number of alphabets which will be serviceable in the drawing of titles, and which are specimens of the type-founder's art, procured for, though not designed with especial reference to, this work.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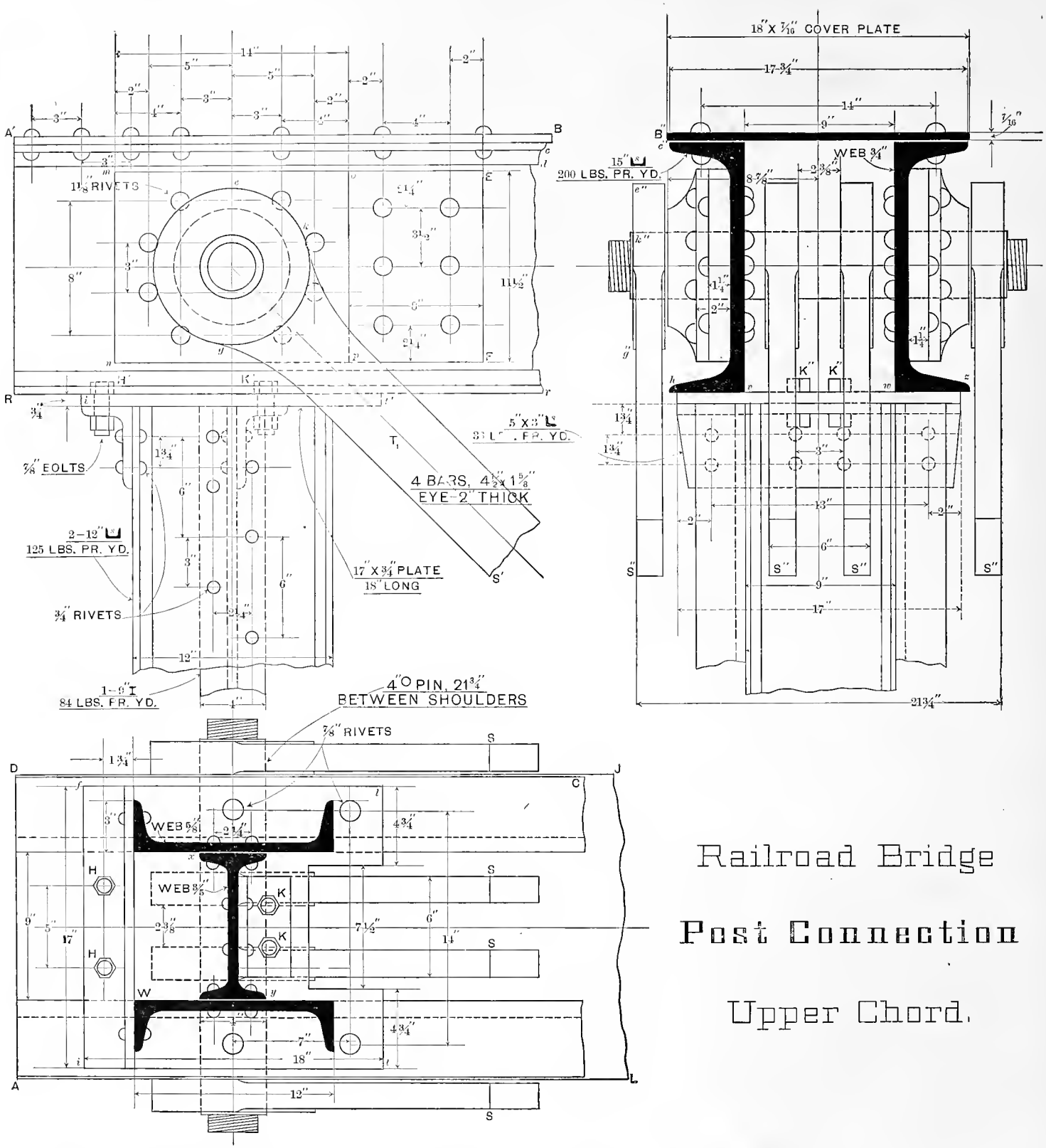
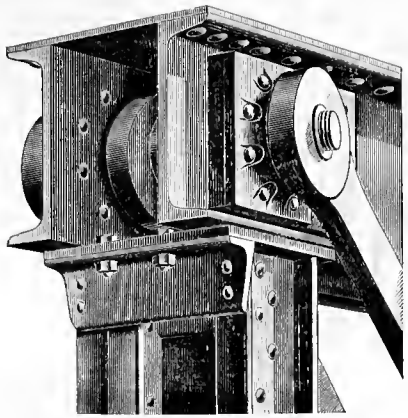


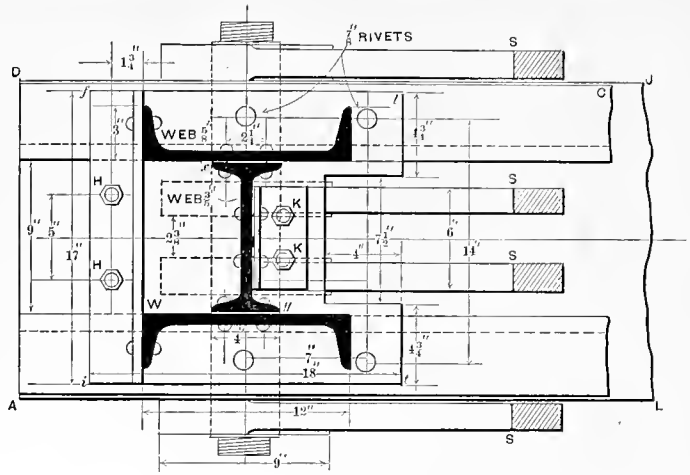
Railroad Bridge

Post Connection

Upper Chord.

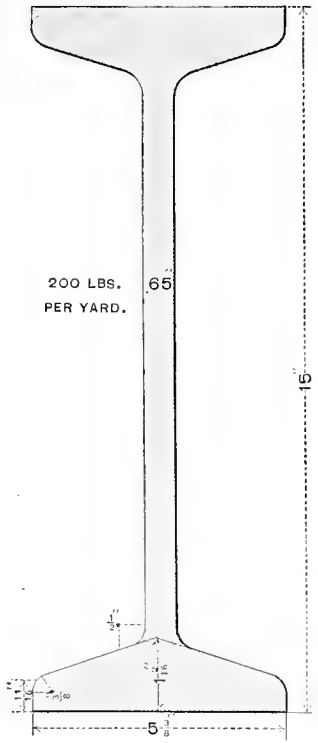
(In drawing the above the student will make *horizontal*, not slanting dividing lines in all fractions. Centre and dimension lines in continuous red lines, unless for blue printing, in which case all lines will be black.)

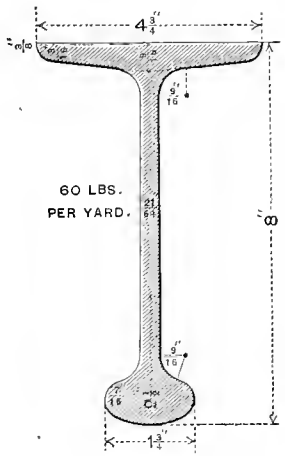




15" I-BEAM.

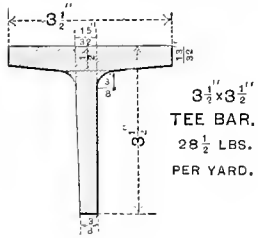
200 LBS.
PER YARD.



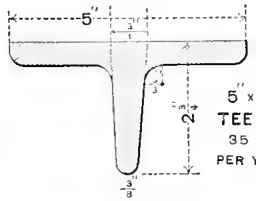


60 LBS.
PER YARD.

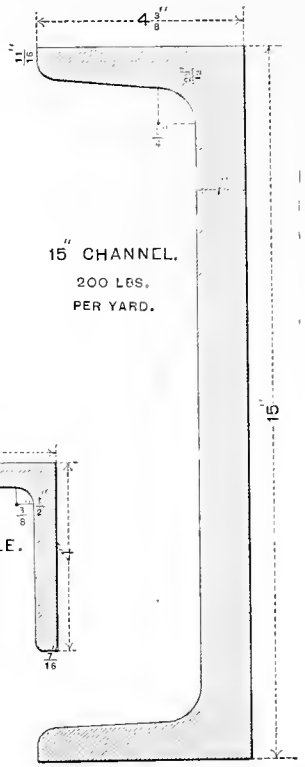
DECK BEAM



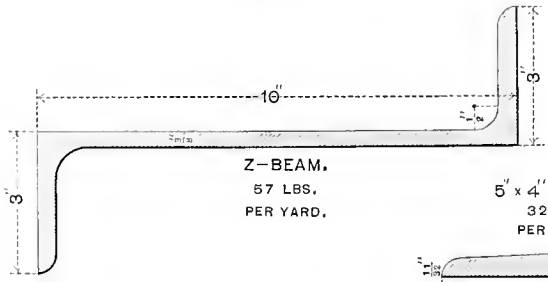
3½"x3½"
TEE BAR.
28½ LBS.
PER YARD.



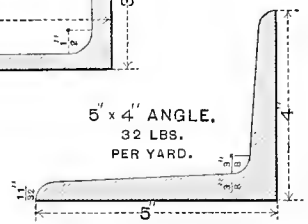
5" x 2¾"
TEE BAR.
35 LBS.
PER YARD.



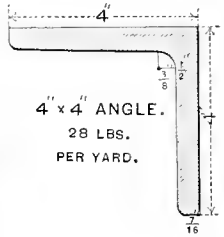
15" CHANNEL.
200 LBS.
PER YARD.



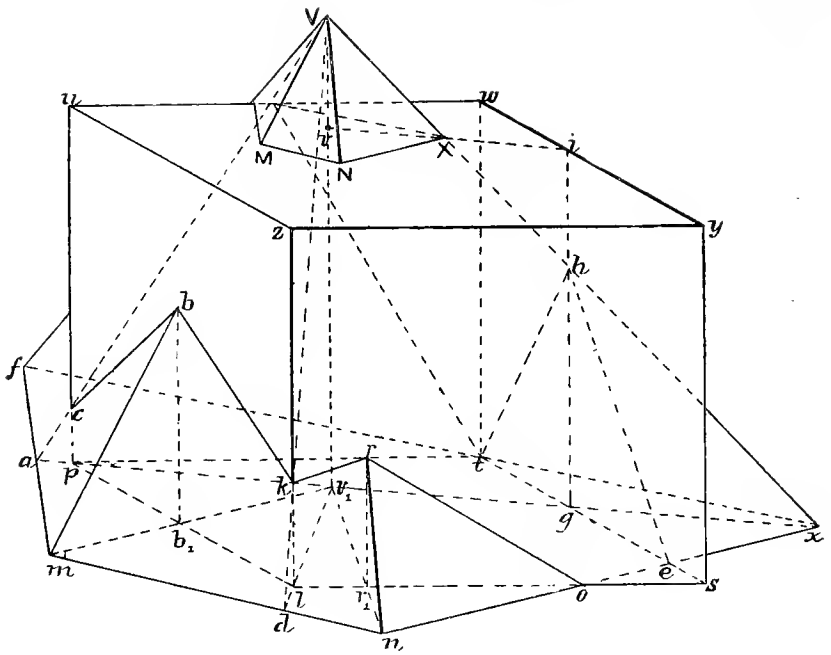
Z-BEAM.
57 LBS.
PER YARD.



5" x 4" ANGLE.
32 LBS.
PER YARD.



4" x 4" ANGLE.
28 LBS.
PER YARD.


V
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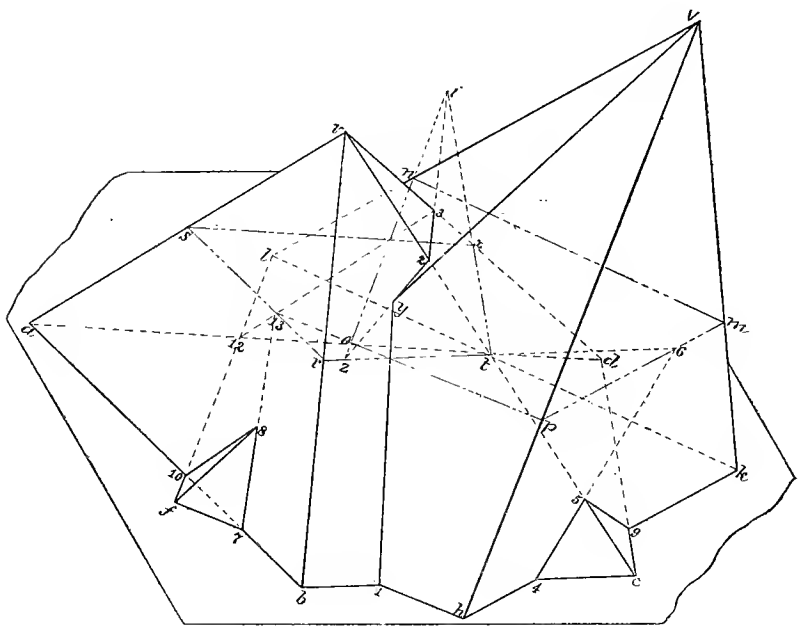
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No. 1.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 2.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & 1 2 3 4 5 6 7 8 9 0 , .

No. 3.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 4.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z & , .

No. 5.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 6.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 7.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 8.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 9.

A B C D E F G H I J K L M N O P Q R S T U V W
X Y Z & a b c d e f g h i j k l m n o p q r s t u v w x y z.,
1 2 3 4 5 6 7 8 9 0

No. 10.

A B C D E F G H I J K L M N O P Q R S T U V W X
Y Z & a b c d e f g h i j k l m n o p q r s t u v w
x y z . , 1 2 3 4 5 6 7 8 9 0

No. 11.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0

No. 12.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z., 1 2 3 4 5 6 7 8 9 0

No. 13.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n o
p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 14.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n o p
q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 15.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n o
p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 16.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & a b c d e f g h i j k l m n o p q r s t
u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 17.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & a b c d e f g h i j k l
m n o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 18.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 19.

A B C D E F G H I J K L M N O P Q R S T U
V W X Y Z & a b c d e f g h i j k l m n o p q r s
t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 20.

A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z & a b c
d e f g h i j k l m n o p q r s t u
v w x y z 1 2 3 4 5 6 7 8 9 0

No. 21.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0 , .

No. 22.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 23.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & a b c d e f g h i j k l m n
o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 24.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 25.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
& 1 2 3 4 5 6 7 8 9 0 , .

No. 26.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z . , 1 2 3 4 5 6 7 8 9 0

No. 27.

A B C D E F G H I J K L M N
O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0 , .

28.

A B C D E F G H I J K L M N O P Q R S T U V W
X Y Z & a b c d e f g h i j k l m n o p q r s t u v w
x y z . , 1 2 3 4 5 6 7 8 9 0

No. 29.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n
o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 30.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 31.

A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z & a b c d
e f g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 0

No. 32.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & . , 1 2 3 4 5 6 7 8 9 0

No. 33.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & a b c d e f g h i j k l m n
o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 34.

𝔄 𝔅 ℭ 𝔇 𝔈 𝔉 𝔊 ℌ ℑ 𝔎 𝔏 𝔐 𝔑 𝔒 𝔓 𝔔 ℜ 𝔖
𝔗 𝔘 𝔙 𝔚 𝔛 𝔜 ℨ & 𝔞 𝔟 𝔠 𝔡 𝔢 𝔣 𝔤 𝔥 𝔦 𝔧 𝔨 𝔩 𝔪 𝔫
𝔬 𝔭 𝔮 𝔯 𝔰 𝔱 𝔲 𝔳 𝔴 𝔵 𝔶 𝔷 1 2 3 4 5 6 7 8 9 0

No. 35.

A B C D E F G H I J K L
M N O P Q R S T U V W
X Y Z & a b c d e f g h i j k l
m n o p q r s t u v w x y z.,
1 2 3 4 5 6 7 8 9 0

No. 36.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 37.

𝔄 𝔅 ℭ 𝔇 𝔈 𝔉 𝔊 ℌ ℑ 𝔍 𝔎 𝔏 𝔐 𝔑 𝔒 𝔓 𝔔 ℜ 𝔖
𝔗 𝔘 𝔙 𝔚 𝔛 𝔜 ℨ & 𝔞 𝔟 𝔠 𝔡 𝔢 𝔣 𝔤 𝔥 𝔦 𝔧 𝔨 𝔩 𝔪
𝔫 𝔬 𝔭 𝔮 𝔯 𝔰 𝔱 𝔲 𝔳 𝔴 𝔵 𝔶 𝔷 ., 1 2 3 4 5 6 7 8 9 0

No. 38.

A A A B B C D D D E E F G H H H
I J J K L M N N N O O P Q R S T U
V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 39.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z & a b c d e f
g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0

No. 40.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z & a b c d e f
g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0

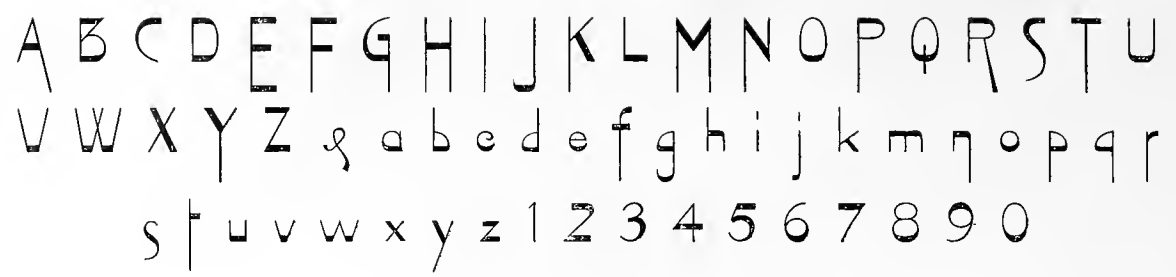
No. 41.

A B C D E F G H I J K L M N O P Q R
T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w
x y z 1 2 3 4 5 6 7 8 9 0

No. 42.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

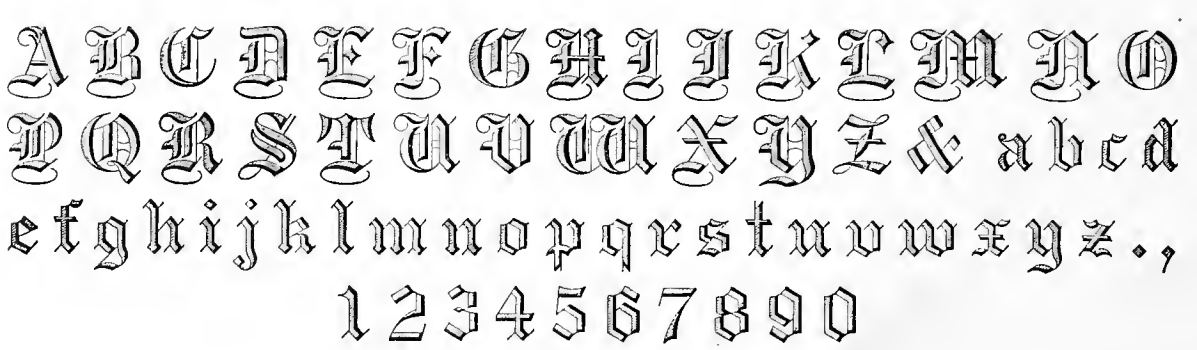
No. 43.



No. 44.

No. 45.

No. 46.



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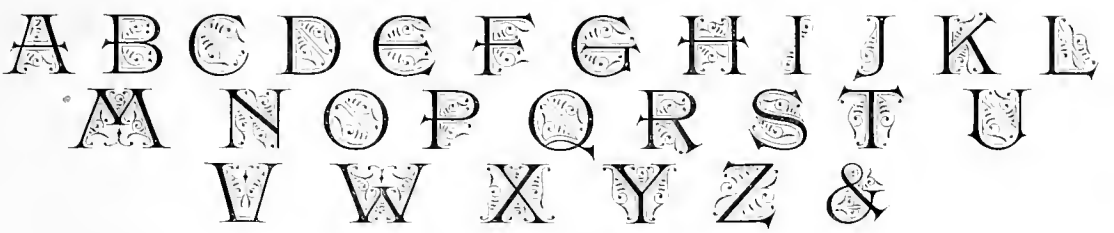
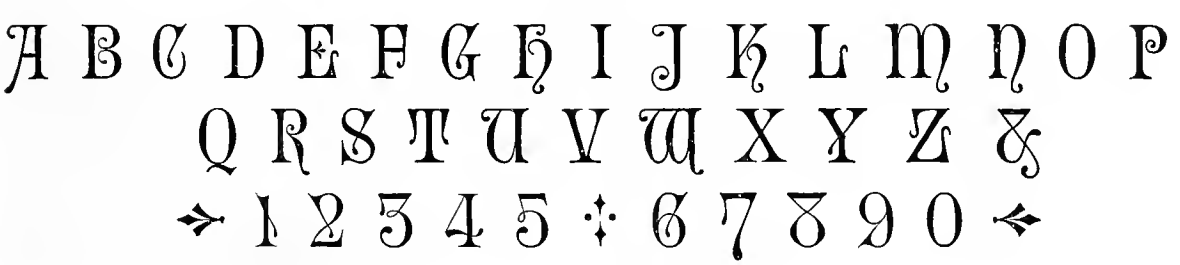


No. 48.

No. 49.



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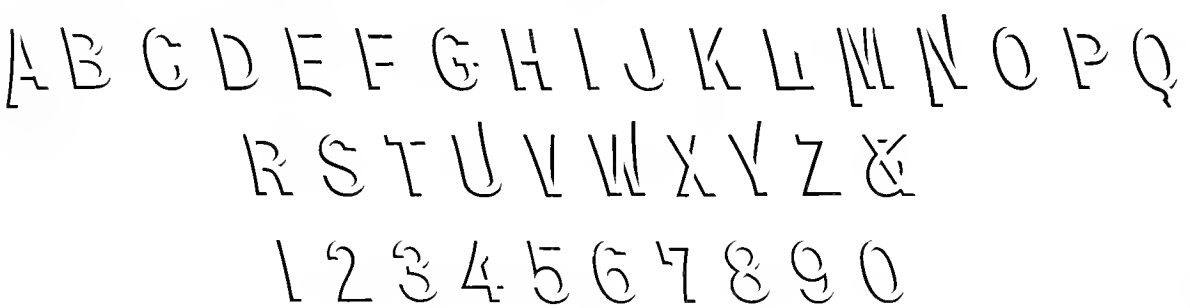
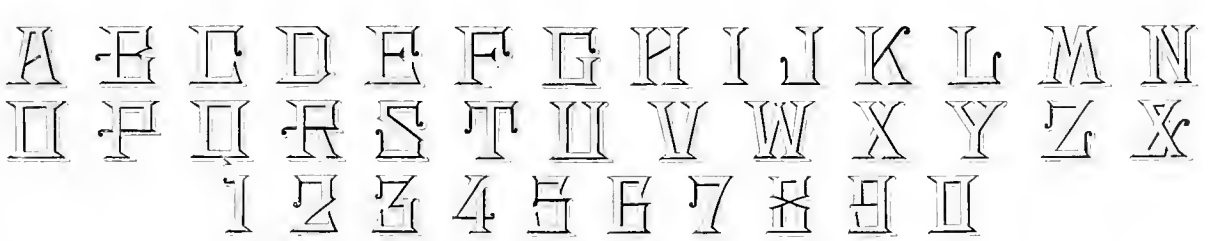


No. 51.



No. 52.

A B C D E F G H I J K L M N O P Q R S T U
V W X Y Z & a b c d e f g h i j k l m n o p q r
s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 53.

A B C D E F G H I J K L M N O P Q R
S T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0 , .

No. 54.

A B C D E F G H I J K L M N O P Q R
S T U V W X Y Z & a b c d e f g h i j k l
m n o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 55.

A B C D E F G H I J K L M N
O P Q R R T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 56.

ABCDEFGHIJKLMNOPQRSTUVW
XYZ&abcdefghijklmnopqrstuvw
xyz.,1234567890

No. 57.

ABCDEFGHIJK
LMNOPQRSTU
VWXYZ&1234 5
67890,.

No. 58.

ABCDEFGHIJKLMNOPQRS
TUVWXYZ&abcdefghijk
lmnopqrstuvwxyz
1235567890

No. 59.

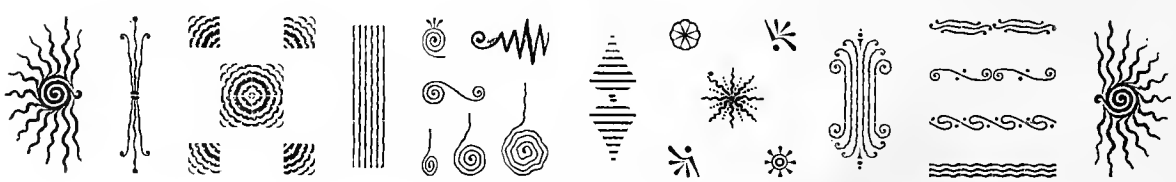
A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z & 1 2 3
4 5 6 7 8 9 0 , .

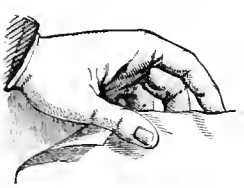
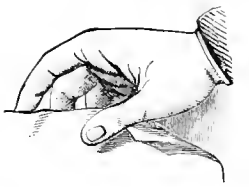
No. 60.

A B C D E F G H I J K
L M N O P Q R S T U V
W X Y Z

No. 61.

A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z a b c d
e f g h i j k l m n o p q r s t u v w x y z . ,
1 2 3 4 5 6 7 8 9 0



No. 62.

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O P Q R S T U

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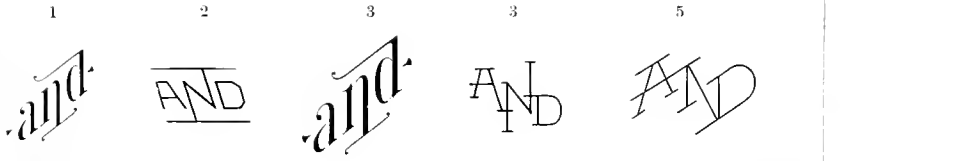

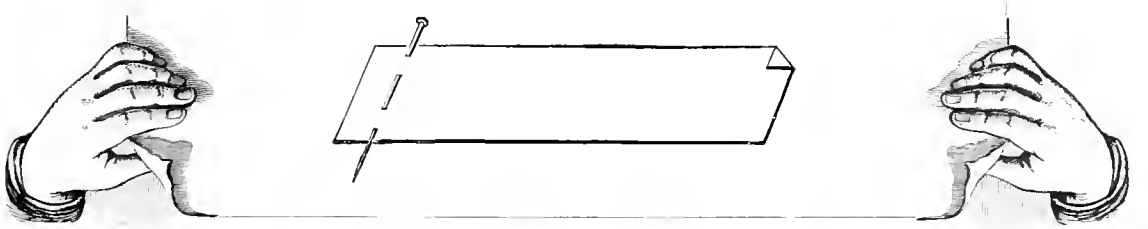

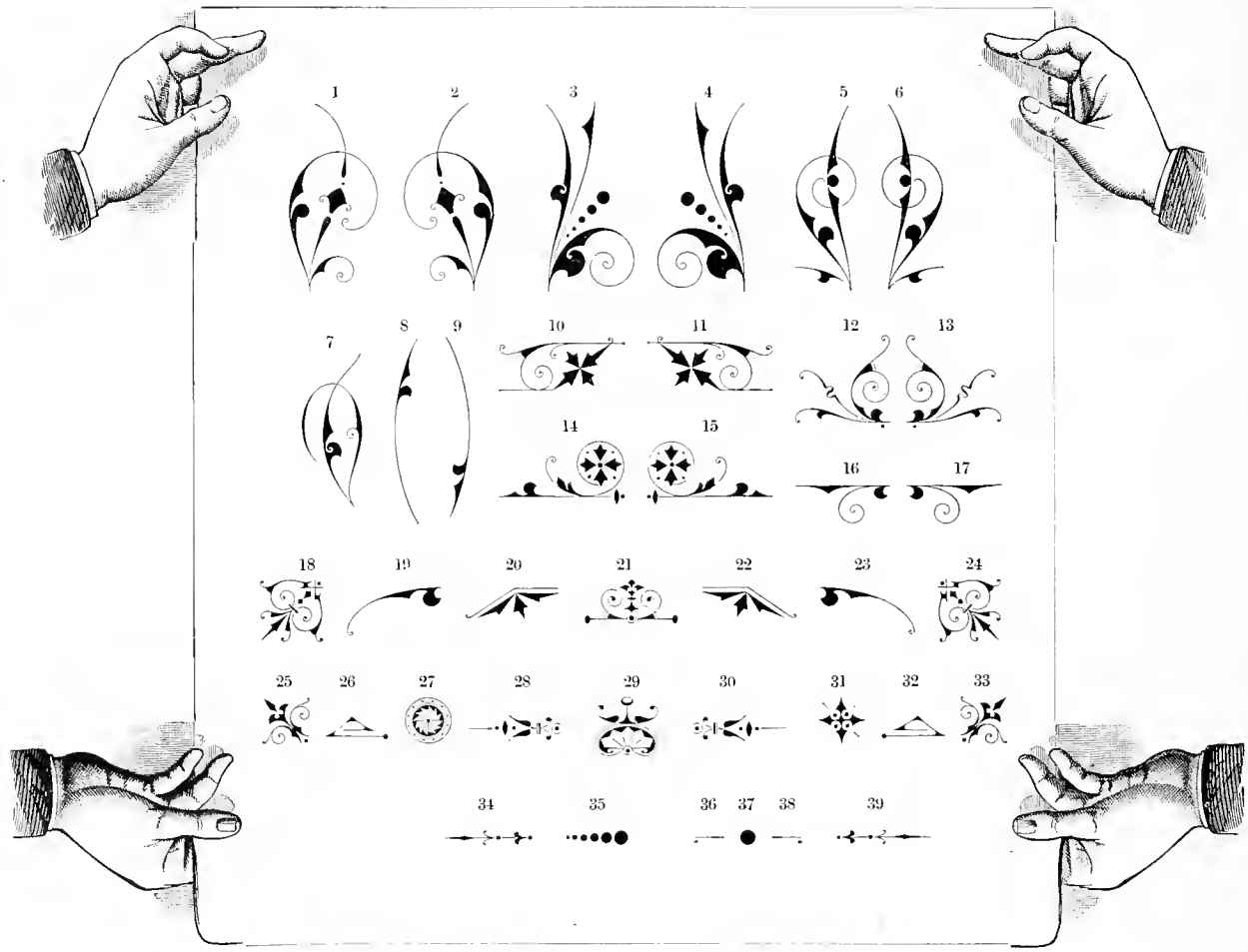
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